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Classifier Toposes & Positive Logic

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Forty years ago, Grothendieck formulated the general program of Classifier Toposes, and his students developed and applied them to analytic spaces and to other fields. The fundamental results of that program have been often helpful in understanding a particular topos, to make explicit a theory that it classifies. But there are several open problems of a basic nature that still need to be clarified.

Any $\mathcal{U}$-Topos $\mathcal{E}$ can be described or constructed by any one of several methods; in addition to the geometrical methods, there is a logical one, which in turn gives unexpected geometric significance to general logical theories; consider a kind of $\mathbb{T}$ of structure that can be interpreted in any $\mathcal{U}$-Topos (like the structure of group or poset) and such that for any morphism $f : \mathcal{E}' \to \mathcal{E}$ of $\mathcal{U}$-Toposes and for any structure A of kind $\mathbb{T}$ in $\mathcal{E}$, f^*A is also a structure of kind $\mathbb{T}$ in $\mathcal{E}'$. (Here f^* is any left adjoint that preserves finite limits). Often there is a $\mathcal{U}$-Topos $\mathcal{U}(\mathbb{T})$ containing a $\mathbb{T}$-structure A_0 with the universal property that for any $\mathbb{T}$-structure A in any $\mathcal{U}$-Topos $\mathcal{E}$ there is an essentially unique morphism

$$\phi : \mathcal{E} \to \mathcal{U}(\mathbb{T}) \text{ for which } \phi^* A_0 = A.$$

The strong restriction on the kind of structure amounts roughly to 'being definable in terms of colimits and finite limits', as could be guessed from the kind of condition on the test morphisms f, but the actual result is part of a complex of completeness theorems. For common shapes $\mathbb{T}$ of structure, the classifier toposes $\mathcal{U}(\mathbb{T})$ turn out simply to consist of all presheaves in $\mathcal{U}$ on some small $\mathcal{U}$-category $\mathbf{C}$: for example, the classifier for mere unstructured objects turns out to consist of $\mathcal{U}$-functors on $\mathcal{U}_1$, the full category of finite $\mathcal{U}$-sets, while $\mathcal{U}$-functors on $\mathcal{U}_1^{\mathrm{op}}$ instead classify Boolean algebras in any $\mathcal{E}$; the topos of simplicial $\mathcal{U}$-sets (defined in 1950 by Eilenberg & Zilber, and massively applied still today in homotopy theory) classifies total orders with distinct end points.

Grothendieck was able 'without logic' to apply these ideas to hundreds of examples, in the sense that for each kind of structure that came up, he was able to discern whether it was expressible by colimits and finite limits, without going through the reduction of the description of such structures to relational systems and the reduction of coverings to disjunctive/existential axioms. The role of specified coverings (Grothendieck 'topologies') becomes especially relevant for those toposes which are not of the form 'all presheaves', but expressible only as subtoposes of such, (because of insufficiency of projectives).

There is of course a huge amount of inter-definability in the sense that $\mathbf{C}$-shaped structures restricted by certain disjunctive/existential conditions J may turn out to be equivalent (in all $\mathcal{U}$-toposes even) to $\mathbf{C}'$-shaped structures restricted by J'.

However, judiciously-chosen presentations by such a pair $\mathbf{C}$, J, are often indispensable in calculating examples. Moreover, description of a structure as a system of maps (sometimes mathematically the most direct) can always be reformulated in a standard way as a description in terms of a system of subobjects ('relational system') and thus a variant of the first order theories of predicate logic can be applied as a powerful technology for manipulating such presentations. The variant (called 'positive logic', 'geometric logic', 'dynamical logic' by different authors) is a system that is actually often used in traditional model theory without giving it the status (which would seem appropriate in view of the present considerations) of 'the basic' logic: There is in general no implication operator definable on formulas, nor universal quantification, so instead of a class of theorems, there is a class of pairs of formulas, the first of which entails the second, and these pairs of formulas, rather than being restricted to sentences, have free variables, (but the same variables in each of the pair). Existential quantification (characterized in 1963 as a left adjoint) does apply to formulas; moreover, the formulas with any given free variables, are closed under disjunction, conjunction, and include special formulas called true, false.

In an interpretation, entailment corresponds to inclusion between subobjects of a given object. Even though formulas in general do not have negations, we can assert that a given formula F is actually false, i.e. denotes the empty subobject, by simply requiring that $\langle F, \text{false}\rangle$ is an entailment pair. To present every subtopos of a given topos may require infinite disjunctions (though not infinite conjunctions); of course, only when finite coverings suffice for $\mathbb{T}$, are we guaranteed 'completeness' in the sense that there are enough $\mathbb{T}$-structures in $\mathcal{U}$ itself.

The restriction to positive logic is appropriate because even though negation, universal quantification, and implication, do indeed have unique interpretations in any fixed topos $\mathcal{E}$, they are not preserved by the appropriate changes f^* of topos. Of course, if one is only interested in structures valued in Boolean toposes (such as Boolean-valued models of $\mathcal{U}$) positivity is no real restriction, because any occurrence of a negated formula $\neg F$ in an axiom can be replaced by a new atomic formula G and the two axioms

$$F \wedge G = \text{false},\ \ F \vee G = \text{true},$$

adjoined. (That procedure might change the natural notion of morphism of models.) In the Stone space of a Boolean algebra, not every open set is clopen; this exemplifies the fact that the classifying topos of a Boolean theory is typically not itself Boolean.

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