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Category Theory and the Foundation of Mathematics *

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I. Foundations as a Branch of Mathematics: the topos of Cantorian Kardinalen as a base for Pedagogy and the topos of differentiable Mengen as a base for Geometry.

A. Some aspects of the concept of category theory which are often not sufficiently appreciated.

There are two things, I guess, the first is that every map has a definite domain and codomain and that two maps can be composed iff the codomain of the first equals the domain of the second. These are principles which were found through decades of arduous struggle in geometry, in algebra and in analysis. It might seem that it might be more convenient to ignore these things or to somehow generalize to get around it, but whenever this is tried produces, to use a technical term from computer science, garbage. That is to say, in particular, it produces a lot of questions which are in reality pseudo questions. For example, if I had

*Key words on the original program: Maxwell's 1871 Program, distributive vs. linear categories, general vs. particular toposes, Euler-Schanuel characteristic and dimension, Carnot thermomechanical quantities, conscious 'bookkeeping".

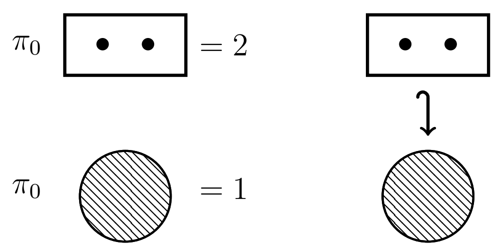
in mind one of the wrong ideas that is still around (one of the attempts to avoid these principles) is that maps themselves are somehow a special case of the notion of a family of "objects"

$$\text{maps } X \to Y \text{ "contained in" families } X \to \mathcal{V}.$$

That is, the idea is you have these maps, but there is no need to specify their codomain. But this is not correct because the fact that a map has a definite codomain was found through struggle, for example, in applying the functor π_0 (the set of components) the result depends crucially on the codomain; of course if one composes a map f with an inclusion map i

$$X \xrightarrow{f} Y \xhookrightarrow{i} Y' \qquad (X \xrightarrow{if} Y')$$

one gets a map if which may have very different properties from f, and even if f is the identity the property that i is a monomorphism is not preserved by the functor π_0.



Even to believe that something like surjectivity is a well-defined property of a map presupposes that the specification of the codomain is part of the specifications of the map. And that it therefore has sense to contrast the image as opposed to the codomain, categories are not fixed any more than the objects are and the means by which they change are functors such as π_0 associating to each Menge its Kardinalen of components. A related dual wrong idea coming from ignoring this principle is that there are things called "terms" which, although they have definite types, do not need to have their variables explicitly declared.

B. Category Theory is the instrument of formal science which unites forever in itself the following necessary features of mathematical thought:

1. Each particular object is completely fixed and definite.

2. Each object can immediately be transformed in determinate ways into other objects.

1 is completely necessary for the whole mathematical enterprise. For example, the group generated by X and Y, subject to the relation $X^2Y = YX$ is completely fixed and definite, or the three sphere etc.

2 is a constantly used method of mathematical investigation; let us see what happens if we impose one further relation, obtaining a new group, or let us see what happens if we move in this direction etc.

It's the failure to have a formal instrument that unites in itself both these necessary aspects, which makes possible thousands of useless books. For example, there is the misconception that accepting 1) makes 2) impossible. Or that universal transformability means that nothing is definite and comprehensible. We could even list these as two typical axioms of anti-science

$$\begin{aligned} 1 &\Rightarrow \neg 2 \\ 2 &\Rightarrow \neg 1. \end{aligned}$$

The first axiom could be based on some experience. We are now speaking about a not very well-defined category, the category of sciences. Any science does have a determinate object of study and high technical competence must be achieved through particular experience, i.e. 1). But then $\neg 2$ "results of science are trade secrets", trade secrets which on the one hand are only available to those who have this experience which can then be repeated in words with less foundation in the object of study and little particular experience. Some initial technical competence, for example, in category theory seems to give us a natural propensity and capacity to look even at ourselves more closely, which those who are committed to trade secrets are terrified to do.

In a similar spirit, we arrive at the notion of 2-categories (where the notion of adjoint comes into play) to which we will return.

Consequences of the other axiom for anti-science $2 \Rightarrow \neg 1$ is a kind of desperate ecclecticism: if one only could combine fragments of linguistics, anthropology, mathematics, computer science, then one perhaps could arrive at an instrument even more reliable than that of

Gentile and Goebbels for freezing everything.

As suggested above, concept of category in itself is already of some use for clarifying sciences in general and mathematics in particular. One of the ancient meanings of "mathematics" is "the teachable". One aspect of science is transforming the thoughts which result from experience, to make them accessible to the larger public, for example children. The instrument of adjoints makes it possible to make explicit more of the steps, than is usually done in explanations. To give an example of this, bookkeepers, at least in the old days, were very good at mental addition. This was in part a trade secret, since it cannot be taught reliably because one had not made explicit all the laws. To teach reliably one must make explicit all the laws, not only the commutative law, associative law and the standardized names. Mere repetition does not necessarily improve the probability of successful explanation.

There are certain laws in mental adition:

$$\begin{aligned} 23 + 35 &= (20 + 3) + (30 + 5) \\ &= (20 + 30) + (3 + 5) \\ \text{Mac Lane middle four} & \\ &= 50 + 8 = 58. \end{aligned}$$

A different example

$$\begin{aligned} 17 + 8 &= (10 + 7) + 8 \\ &= 10 + (7 + (3 + 5)) \\ &= 10 + ((7 + 3) + 5) \\ &= 10 + 10 + 5 \\ &= 25, \end{aligned}$$

using associative law, commutative law, or in -"middle four" law.

We explained **something** in the first example, but not everything in the second example. "Why didn't you write $8 = 1 + 7$ – which trade secret did you use"? "I believe I will never understand mathematics". We can explain it by adjointness. There is a category around, a "trade secret" (Mac Lane's book!) a "comma category" (I disclaim any responsibility for this word) – take (comma categories of commutative monoid): $a \xrightarrow{x} b$ is a **morphism** if

$a = x + b$; composition

$$a \xrightarrow{x} b \xrightarrow{y} c, \qquad a \xrightarrow{x+y} c$$

A/c in Grothendieck's notation.

Consider multiplication by ten. It is easily verified that this is a functor, because if X substantiates that $a \geq b$, then $10X$ substantiates that $10a$ is greater than or equal to $10b$. This functor has both a right and left adjoint so that together we get an adjoint graph, meaning that moreover $\alpha(10x) = x$; that α is the left adjoint means that

$$\frac{\alpha(x) \geq y}{x \geq 10y}$$

so that, for example $\alpha(43) = 4$; so that the composite involving alpha when applied to 43 gives 40; by contrast the composite involving beta when applied to 43 gives 50.

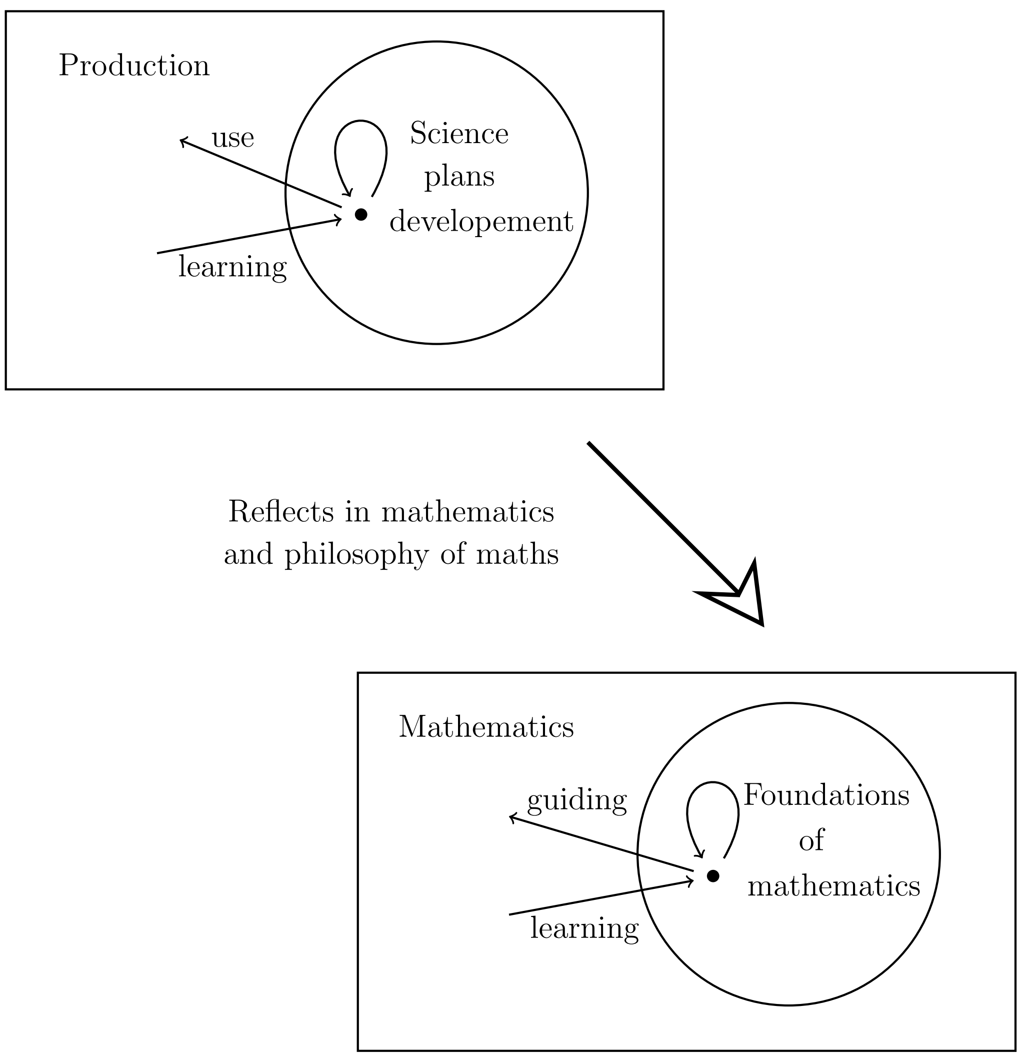
In other words, the secret of the above calculation involved looking for the multiple of 10 which is nearest, which is a special kind of adjointness. Then, of course, we must deal with the notion of difference which is another adjoint. In Grassmann's terminology an operation such as plus which puts things together is called a synthetic operation, and an operation which attempts to invert it is an analytic operation. Usually, the analytic and synthetic operations are not actually inverse so that without the notion of adjointness the relation between the two remained vague though useful. Schroeder gave the example of conjunction and implication or dually disjunction and logical subtraction, which are obviously not simply inverse. However, both these examples, as well as our present example of subtraction of non-negative quantities, are examples where the analytical operation is uniquely determined by the synthetic one because they are adjoint and in fact together constitute a so-called closed category.

Note the double usage of the word "deduction" which is used both for logical deduction and also for the deductions from your pay check which is an instance of truncated subtraction;

these two usages of the word are revealed as two examples of closed categories

$$\frac{a+b \geq c}{a \geq c-b}$$

The final point which needs to be made explicit in order to complete the above calculation is that the closed category in question has the following special property: if $b-c=0$ then $b+(c-b)=c$. Here the last equality means that the "evaluation map" reduces to an isomorphism when the two objects in question have the hypothesized relation $c \geq b$. While we don't invoke all these laws everytime we make a calculation, they can be reliably referred to once they have been made explicit.



II. Mathematics as the Study of the Quantitative Motion of Bodies in Space; Grassmann and Dialectics, and the need of Objective Logic, as well as Subjective

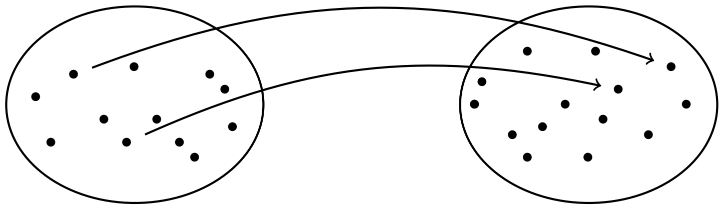
I included in my list of possible subtitles a list of keywords which did not get printed. One of these was “Maxwell’s 1871 program”. This program was not presented here, but at one of the early meetings of the London Mathematical Society. The program was to classify physical quantities. As I understand it, this was just after he had made his fundamental discoveries in electromagnetic theory and had published them in the form of research articles, but was then engaged in writing them in form of a book. One can infer from that that he was concerned with pedagogical questions. Therefore precise terminology and clear conceptual structure for the purpose of explaining electro-magnetism was a central concern. Therefore he proposed to the London Mathematical Society a program of classifying physical quantities, made several interesting comments and made definite progress toward carrying out that program. However, in my view, that highly laudable program is still not carried out completely. We are nowadays told that all quantities are the same (real numbers). Thus one direction of these lectures is to make further suggestions for advancing that program.

The pedagogical concern was the reason for my 2nd published paper in 1964. My first job was to teach calculus to first year students of Reed College in Portland, Oregon. I was told on arrival that this was a college of highly gifted and qualified students and that the structure of the curriculum in the Mathematics Department was such that the rules of calculus should be learned in the second year after a first year devoted to the professor’s interpretation of what constitutes the foundation of calculus. The typical response to that challenge was to spend most of the first year in a careful construction of the real number system from Peano’s postulates, making explicit at each stage what equivalence relations are involved etc, no doubt an educational process. However, my response was that I would like to have a rigorous framework within which to prove the rules of calculus. At first I considered using Zermelo-Fraenkel set theory which I had learned from a year’s stay in Berkeley, but I estimated that to explain this Zermelo-Fraenkel set theory, would for first year students probably requiere most of the year, leaving very little time for discussing calculus.

On the other hand, this structure would be so completely foreign to the actual practice of higher-dimensional calculus, as one learns it in courses on analysis and continuum mechanics. Pondering these questions I formed the idea "Why shouldn't foundations be as much like mathematics as possible? I knew by that time that mathematics was much like category theory, there were topological categories, algebraic categories, and so on, which while all different, nonetheless had a lot in common. So I thought one should discuss the category of sets. I estimated that I could discuss that in two months, so I'd have the rest of the year for discussing calculus. I discussed various constructions in the category of abstract sets, writing down the key ones and calling them axioms. By October 1963 I had 18 axioms. In the next few months I was able to reduce these 18 to 6 axioms and publish the elementary theory of the category of sets in that form at the end of 1964. In the course, I continued the discussion of other categories which seemed to be relevant, the category of metric spaces and distant decreasing maps, the category of boolean algebras, the category of vector spaces. I ended with proofs of the fundamental theorem of calculus and the Leibniz product rule for differentiation. If we view this experience in the light of our current experiences with teaching, it seems quite impossible, but first of all these were indeed bright students and secondly, they did in fact learn it, (some went into math later). In any case, this was the actual origin of concentrating, not on the categories of central interest to mathematics such as differentiable Mengen and abelian categories of modules etc. but rather on the simplest case of no structure, namely the abstract sets; having stated sufficient axioms about the latter, one is then able to deduce the construction of the more richly structured categories, such as metric spaces and boolean algebras, and thus provide a basis for the discussion of such things as the fundamental theorem of calculus.

Much later, I was able again to teach a similar course to under-graduates, when about 5 years ago I requested permission from the local logic committee to teach the third year course in introductory set theory. The opposition was pretty strong, but fortunately I was supported by that fine Cambridge man John Myhill, who thought I should teach the course.

Myhill attended the first few lectures in which I explained that an abstract set is a bag of dots which have no properties, except that they are distinguishable.



That is, the dots are completely indistinguishable and yet completely distinguished. Further, the bag itself has no properties, except "cardinality", i.e. it might or might not be isomorphic to another bag. This notion is extremely abstract and therefore extremely simple. The other ingredient, of course, is that we must consider arbitrary transformations between these objects and in this case "arbitrary" is extremely broad, since there are no properties for the morphisms to preserve.

Myhill told me: "I have seen all this before". He brought me his copy of the Works of Cantor with the note to study a certain passage. He was right. Cantor **had** invented the notion of abstract set, which he called a **cardinal**. The later notion that a cardinal is instead a "canonical" representative of an isomorphism class (e.g. initial ordinal) completely obliterates the fundamental insight of Cantor concerning the contrast between "Mengen" and "Kardinalen".

Several days ago I was again reading this book of Cantor on the train from Milan to Zurich when I noticed two additional historical points that seem to have been submerged by history. Therefore I went to inquiere of the respected set theorist Ernst Specker for further clarifications of these three points. His response was that the points were indeed very interesting, but that also he did not recall their having been discussed. We went together to the library at the ETH and tried very hard to find information but were unsuccessful.

I'll come back to the other two points later, but now let me start by describing the first one in more detail:

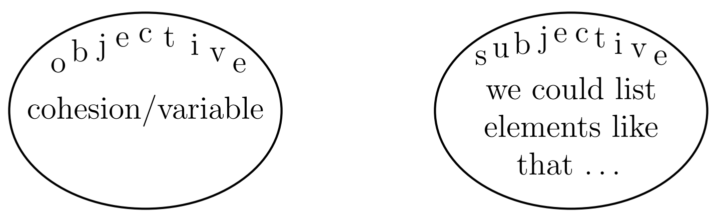
M E N G E N — are like a color TV picture, you can turn down the color and up the contrast, then you have just a bunch of dots, a cardinal.

$\downarrow$

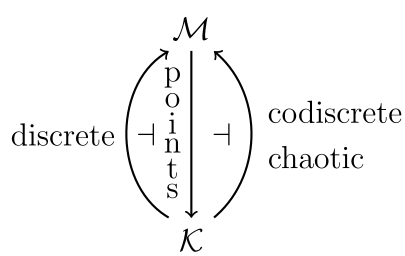
K A R D I N A L S — Kardinals just are these abstract sets.

What are Mengen? They are cohesive/variable sets.

We know this because Cantor started his mathematical life with his feet on the ground. For example, he studied the sets of uniqueness of Fourier series. So he is talking about sets on the line which certainly has some kind of cohesion. One type of a variation involved was also the iteration of the derived set operation. Of course, there are many categories of cohesive sets. There are pure cases of cohesion, pure cases of variability. The passage to the associated cardinal or abstract set of points, abstracts from this cohesiveness and variability. Actually the cohesiveness and variability may have objective aspects (such as the way the parts of the line stick together) and also subjective aspects which model the possible processes by which we get to know the elements (for example, the category of recursive sets).



Both these aspects of cohesiveness/variability are to be abstracted away in the passage to the associated Kardinal; the usual Mathematical word for the latter is the "underlying set of points".



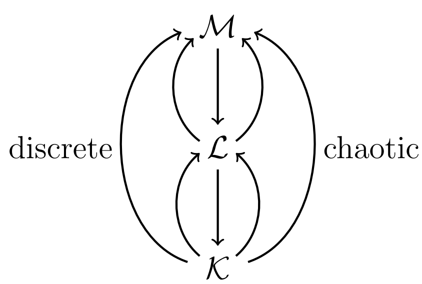
leaving aside the nature of the cohesion and also leaving aside the subjective cohesion. The points form a kardinal. Cantor says "any Kardinal itself also forms a Menge". How can it be that the points are both indistinguishable and yet distinguished? The editor of Cantor's collected works gets into a muddle over this, somehow thinking that this contradiction leads to an inconsistency, and he thus advises us to ignore this part and get on with the main business, which should be Kardinal arithmetic. This has no doubt influenced the students of the great man's difficult works, who have consulted the editor for guidance, not to think too much about this very relationship that we are emphasizing. But more precisely, there are two ways in which a Kardinal itself determines a Menge.

$$\frac{\mathrm{dis}(K) \longrightarrow M \text{ in } \mathcal{M}}{K \longrightarrow \mathrm{pts}(M) \text{ in } \mathcal{K}}$$

determines one the discrete way, namely, the empty case of cohesiveness and variability, namely no cohesiveness. Any map from a discrete space is continuous; more precisely, the above natural bijection reveals the discrete functor as the left adjoint of the points functor. Here "continuous" just means preserving the cohesiveness and variability in a particular sense specified by the category $\mathcal{M}$. In this way we succeed in keeping the points distinct, for if not, there would be some map .... On the other hand, there was something to the idea that since these points have no properties, they are all the "same" in the sense that they all cohere in a trivial, immediate way with each other; this, however, is an intuitive description of the **right** adjoint which associates to every Kardinal a chaotic or codiscrete Menge with the same points

$$\frac{M \longrightarrow \mathrm{chaotic}(K)}{\mathrm{pts}(M) \longrightarrow K.}$$

That is, to map continuously an arbitrary Menge M into a codiscrete one is equivalent to mapping arbitrarily the points of the two. For example, if M were a model of time, i.e. a motion parametrizer, then continuous motion in a codiscrete space is really an arbitrary mapping from the Kardinal of instants of time into the points of the codiscrete space, whereas by contrast, if (as is reasonable in such a picture) we imagine that M is connected, the continuous motions in a discrete space are all constant; in fact, the latter serves as a definition of "connected", in particular, any two points of a codiscrete space can be connected by a continuous motion (provided there are two distinct instants of time). If we compose two canonical maps we get a canonical map from discrete of K to codiscrete of K which upon applying the point functor becomes an isomorphism in $\mathcal{K}$ but which is itself not an isomorphism in $\mathcal{M}$, unless K equals 0 or 1. In fact, the only continuous maps from a codiscrete of K to discrete of K are constants. Frequently this canonical map from discrete to codiscrete is a monomorphism; that is, it is not that the distinct points as such become identified, but rather that the cohesiveness changes, so that on the right hand side the points can be transformed into each other without effort. Thus, to sum up, we have an *adjoint cylinder* (a configuration of categories and functors dual in form to the *adjoint graphs* discussed last time in connection with the numerical example). That is, not only are the three functors adjoint, in a sense explained, but moreover, the two composites at $\mathcal{K}$ are the identity. Such adjoint cylinders I propose as the mathematical model for many instances of the philosophical relation of **unity and identity of opposites** (**UIO**); it may be considered equivalently as a property of the single functor here called "points", because of the essential uniqueness of adjoints. That is, $\mathcal{M}$ unites the discrete and codiscrete; $\mathcal{K}$ is a subobject in two opposite ways of $\mathcal{M}$. But recall that a subobject always means an inclusion; if we ignore the inclusions, we get the identical $\mathcal{K}$. The oppositeness is expressed here uniquely by the two opposite senses of adjointness; it does *not* mean that the two subobjects are "disjoint", since here they overlap in the "truth values" 0 and 1. Now one can go further with this notion of Unity and Identity of Opposites because after having learned basic features of Kardinals we can start looking back at Mengen, i.e. at the mathematics, with some more precise tools in hand. Perhaps Cantor himself intended to do this, since at one point he suggests that the distinction between ponderable matter and ethereal matter might be just a question of Kardinality, a suggestion which (although probably insufficient) does at least show an intention that this abstract machinery should be applied to a pressing problem. At any rate, one finds that there are often further adjoint cylinders in between



with an intermediate category $\mathcal{L}$, less abstract than $\mathcal{K}$, and containing more information about the "real" objects $\mathcal{M}$ but simpler than $\mathcal{M}$ and perhaps more amenable to computation. Let me give an example

$$\mathbf{\Delta}_1 = \{1, \partial_0, \partial_1\} \qquad\qquad \partial_i \partial_j = \partial_i \quad i, j = 0, 1,$$

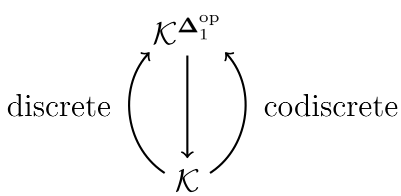
this is a three-element monoid and we can consider the category of right $\mathbf{\Delta}_1$-sets $\mathcal{K}^{\mathbf{\Delta}_1^{\mathrm{op}}}$ also known as

- right actions of $\mathbf{\Delta}_1$ on cardinals,
- abstract applications of $\mathbf{\Delta}_1$,
- presheaves on $\mathbf{\Delta}_1$.

In this case the objects are also known as **reflexive, directed graphs** and the Cayley-Dedekind-Yoneda embedding produces the generic example I which has the internal picture ((explanation of why they are graphs))

$$I = \boxed{\bullet \longrightarrow \bullet}$$

Since I in general signifies a little category acting on itself, and in this case it has three elements with the indicated action. There is then an adjoint cylinder



in which the points functor omits all the connecting arrows from a given graph and in which adjointness forces the codiscrete graph on a given set of points to have exactly one arrow connecting each given ordered pair of points.

The points functor is actually representable by the graph which has a single point and only the degenerate arrow at that point, as is also seen as a special case of the left adjointness of the discrete graph. If I have an interesting graph X with some information in it, and map its points into the points of the codiscrete graph, then that map can be extended uniquely to a graph morphism from X itself because every arrow in X has exactly one place to go in the codiscrete graph.

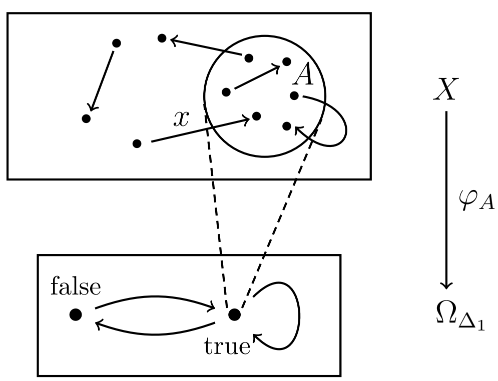
Such an adjoint cylinder reminds me of the famous slogan “ultraleft is right in essence” because the codiscrete objects can be mapped around quite arbitrarily and yet their category in itself is identical with the discrete ones, in which no motion is possible.

This example is worthy of a little more discussion, since both $\mathcal{K}$ and $\mathcal{K}^{\mathbf{\Delta}_1^{\text{op}}}$ are toposes. $\mathcal{K}$ forms a Boolean topos with truth value object

$$\boxed{\Omega_{\mathcal{K}} = 2 = 1 + 1}$$

This Booleaness does not necessarily imply that $\mathcal{K}$ are totally abstract sets, since one way of summarizing Galois' discovery is to say that when $\mathcal{M}$ is algebraic geometry over a non-algebraically closed field, then the natural base topos $\mathcal{K}$ is a Boolean topos involving actions of the Galois group. On the other hand, the truth value object for the reflexive graphs turns out to be a five-element graph which cannot be disconnected as a sum in any non-trivial way.

Given any subgraph A of any graph X then the arrows of X are partitioned into 5 kinds relative to A: those which are totally or truly in A, those which are completely outside A, those which enter A (as little x in the picture) those which leave A, and those which make an excursion from A only to return. The truth value object is in some sense the pure case where all these possibilities are realized by taking A to be the single point true.



But on the other hand, for an arbitrary configuration A and an arbitrary X there is a unique graph morphism φ_A which maps A to "true" known as the characteristic morphism of A. Ω also has a Heyting algebra structure given by graph morphisms. We can start measuring things using truth values. Consider, for example, the canonical map from any X to the codiscrete graph on the points of X; its image is a subgraph of the chaotic graph which therefore has a characteristic map to Ω.

$$
\begin{array}{ccccccc}
X & \xrightarrow{\text{canonical}} & & \text{Chaotic}(\text{pts}(X)) & & \xrightarrow{\varphi_{A_X}} & \Omega \\
& \searrow & & \nearrow \quad \searrow & & \nearrow & \\
& & A_X & & \Omega_1 & &
\end{array}
$$

Since this particular subobject contains all points, the characteristic map actually factors through the subobject Ω_1 consisting only of the loop at "true" and true itself. What this measures is the following: An arrow of the chaotic graph is really just a pair of points, but it goes to "true" iff there exists an arrow in X between those points. Otherwise it goes to the loop.

Are there adjoint cylinders between abstract sets and graphs? No, because graphs are one-dimensional. On the other hand, the graphs themselves may very well occur between abstract sets and some more complex category $\mathcal{M}$. ((FWL: Not in the lecture, but belongs here: The crucial ingredient for this to occur is the discovery in $\mathcal{M}$ of an object T which has precisely 2 points and precisely three endomaps in $\mathcal{M}$, for then we can use T to begin a less trivial analysis of the arbitrary objects M by defining a T edge of M to be any map $T \to M$ or, in other words, a point of the function space M^T. In this way we define a functor from $\mathcal{M}$ to $\mathcal{K}^{\Delta_1^{\text{op}}}$ which in many cases will have both left and right adjoints forming an adjoint cylinder, thus decomposing the cylinder structure of Mengen over Kardinal as a stack of two adjoint cylinders and giving upper and lower bounds on an object M once we know its graph.)) As suggested, this analysis in terms of stacks of cylinders has a lot to do with dimensionality of the objects of $\mathcal{M}$, and hence to obtain a specific example, let us consider $\mathcal{M} = \mathcal{K}^{\Delta_2^{\text{op}}}$ the category of triangulated surfaces/$\mathcal{K}$. Here Δ_2 is the 10-element monoid of all order-preserving endomaps of a three-element totally ordered set. The Yoneda embedding gives us a single object that is to be thought of as a solid triangle with its boundary structure. It turns out that the truth-value object is a Heyting algebra with 19 elements which is also a triangulated surface with only two of the elements degenerated all the way to points. The structure of an arbitrary triangulated surface X is determined by the abstract set of all morphisms or figures in X from the solid triangle, structured by the incidence relations given by the right action of the 10 endomaps of the solid triangle; since the arbitrary morphisms from $X \to Y$ preserve this action, they also preserve these incidence relations.

I would like to have a model of the surface that consists of the 19 values of superficial truth - I think it looks like the Sydney Opera house. We arrived at the number 19 by counting the sub-objects of the generic triangle, with the graphs we can arrive at the 5 truth-values by considering all subgraphs of the generic arrow (note that then there is precisely one element A among the 5 that does not satisfy the equation $A = \neg\neg A$; in that sense it is a kind of simplest possible example of a non-Boolean Heyting algebra. This element, considered as a sub-graph of an arrow is just the discrete one of two points, whereas considered as an element of a truth-value **graph** it is the loop!.

Applause

Mayberry: The idea of a set which you attribute to Cantor is actually due to Plato. What is actually your intuitive idea of cohesion and variablility? Maybe I'm asking you to do something stupid, but the simplest case I could think of would be topological spaces, how do you fit this idea of composition and variation into this picture?

Lawvere: Remember that we started with all spaces, we are not saying at all that the zeroth step in analysis explains the cohesiveness that we attribute to the arbitrary topological space. Only two extreme cases of it.

Mayberry: I see that, I had the idea that you were thinking that the points come originally embedded in some kind of Nexus and that the passage to the underlying Kardinal is precisely the forgetting of this Nexus. Is that sort of the idea?

Lawvere: Yes that's true. But perhaps I'm giving you the wrong impression. I've only described the zeroth step of a process ((FWL added: and that at the end of the lecture began to indicate the first step)).

Bell: But $\mathcal{M}$ is not the category of topological spaces in the usual kind of sense, because that's not a topos.

Lawvere: The category of topological spaces would be an example of my basic construction, but **the simplest examples** are the graphs that do indeed form a topos. Many other examples (such as differentiability) can also be construed as topos, but $\mathcal{M}$ might also contain aspects such as real physical motion. I think that $\mathcal{M}$ in general may not initially be known to consist at all of mathematical structures in the usual sense, but through studying we might arrive at a category $\mathcal{C}$ of mathematical structures, such that we can construct a lifted functor, thus more closely approximating $\mathcal{M}$ mathematically. What I called algebraic structure in my 1963 thesis is an instance of this lifting process, the kind of mathematical structure that $\mathcal{C}$ has, is related to what Hegel called 'the abstract general associated to a particular".

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathcal{C} \\ & \searrow & \downarrow \\ & & \mathcal{K} \end{array}$$

If you then take the concrete general associated with the abstract general, what you have is the category of all structures on kardinals which are of the type that is natural for the functor to $\mathcal{K}$. Thus, for example, $\mathcal{C}$ might be the category of topological spaces with the lifted functor from $\mathcal{M}$ showing that among the attributes of the real objects in $\mathcal{M}$, is the possibility of a natural definition of open sets of points. The comparison functor can also be construed as a link between real science and formal science in Grassmann's sense, a link, in fact, more or less determinant.

III. Extensive and Intensive quantities in Logic and Geometry constructed in the Galileo-Cantor-Burnside-Grothendieck spirit

The Maxwell 1871 program (of classifying the physical quantities) has led, for example, to the notion of vector bundle that can be construed as a vector space in a particular topos (in contrast to a general topos). But there are also dimensions built up, say from mass, length, and time, leading for example to forces, rates, fluxes, and densities. (In modern physics there is a tendency to identify some of these via the fixed value for the speed of light). That part of the program is known as dimensional analysis. In its present development it is very useful, but still needs to be incorporated rationally into the whole. There is also a very important distinction between

extensive - and intensive quantities,

an old philosophical distinction that is usually only mentioned in connection with thermodynamics, but which is in fact applicable everywhere in physics, and even in mathematics and logic. Thus the full carrying out of the program should involve making precise all the general relationships between

constant and variable
general and particular
intensive and extensive
dimensional analysis
vectors versus tensors
vectors versus spinors
quantitative and qualitative

in short, a dialectical development of what mathematics is all about.

Grassmann, in the introduction to his 1844 "Ausdehnungslehre", gives a very clear dialectical materialist development of what is space, what is quantity, etc.

For lack of time I will skip over that and give a direct description of two corresponding kinds of categories.

Space

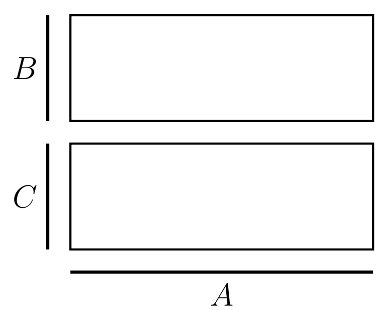
What should space be like? I propose distributive category $\mathbb{D}$.

This is a concept much more general than "topos" that was developed by Schanuel and me several years ago and has found quite a few uses recently, even in computer science.

Distributive category is a category $\mathbb{D}$ that has coproducts denoted by 0 and + and products denoted by 1 and $\times$. In any category where these exist there are canonical maps

$$A \times B + A \times C \xrightarrow{\sim} A \times (B + C)$$
$$0 \xrightarrow{\sim} A \times 0$$

In a distributive category these maps are assumed to be isomorphisms. This does describe one important feature of spatial intuition.



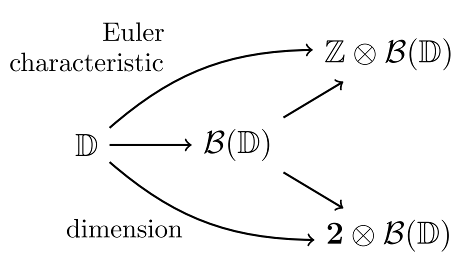
This gives rise to the usual distributive law for certain quantities, if we measure the objects in the spirit of Galileo, Steiner, Cantor, Burnside and Grothendieck: a proper measure of the thing is the thing in itself. For the various partial measurements of magnitude such as length, volume, mass, dimension, cardinality will be attributes of that. This "in itself" refers to the following procedure: Forget the morphisms, except for the isomorphisms, and regard two objects as equivalent if they are isomorphic. The product and coproduct functors become algebraic operations on the resulting "quantities", yielding the Burnside Rig of the distributive category. Here the term Rig is designed, so that if you put in a small n (for negative) you get a Ring, but one should not force n too early. For example, the distributive category of finite Kardinals and maps has as its Burnside Rig the Rig of natural numbers. Given any Rig, we can always tensor it with the integers to get a Ring. For example, if G

is any finite group, then we can consider the topos of finite Kardinals on which G acts and the Burnside Ring of this distributive category is by definition what is called the Burnside Ring of the group G; it reflects important properties of a group, for example, the Ring has no non-trivial idempotents iff the group is solvable, whereas its reduction mod 2 is a Boolean algebra iff the group is of odd order. On the one hand, the algebra of the Burnside Rig reflects important properties of the distributive category, for example, properties of a group as expressed in the category of its permutation representations, but on the other hand, any Rig which arises in this way has the property that its abstract algebra can be greatly aided by going back to the original objects and manipulating them in ways which use the non-isomorphisms. It is this marvellous property which makes cohomology more easily calculable in terms of actual vector bundles.

Tensoring with $\mathbb{Z}$ or $\mathbb{Z}_2$ is not the only kind of useful reduction of the Burnside Rig. A very striking discovery of Schanuel involves the Rig called 2 in which $1 + 1 = 1$. Tensoring a Rig with 2 yields a new Rig in which $X + X = X$, but not necessarily $X^2 = X$. Grassmann characterized logic as the study of quantities which are idempotent in one sense or the other and he also emphasized that extensive quantities differ from abstract quantities in that they have dimension. Schanuel's proposal is that

$$\boxed{\mathbf{2} \otimes \mathcal{B}(\mathbb{D}) = \text{rig of dimensions of } \mathbb{D}}$$

Every object in the original category determines its size which is a quantity in the Burnside Rig $\mathcal{B}(\mathbb{D})$ whose equivalence class in the Rig of dimensions should be called the dimension of the original object. On the other hand, the possibly negative quantity in the Burnside Ring deserves to be called the Euler characteristic of the original object



For many $\mathbb{D}$ the Burnside Ring reduces to $\mathbb{Z}$ itself, so that the Euler characteristic of the original object is just an integer. But in other cases the Euler characteristic may have several components, for example a bounded and unbounded component. That is, the kind of quantity that the Euler characteristic should be, is determined by the actual problem $\mathbb{D}$ and not by an a priori imposition that it must be just an integer for all $\mathbb{D}$. Similarly for dimensions. Again, in simple cases the dimension Rig turns out to be just the set of all symbols δ^n for $n = -\infty, 0, 1, 2, \ldots, \infty$ added and multiplied according to the rules

$$\begin{aligned} \delta^n + \delta^m &= \delta^{\max(n,m)} \\ \delta^n \cdot \delta^m &= \delta^{n+m} \end{aligned}$$

(Often infinity is omitted.) This gives indeed a commutative, associative and distributive system in which addition is idempotent. The exponent $-\infty$ gives the dimension of the empty space, δ itself is a dimension of any one-dimensional space in $\mathbb{D}$, etc. $1 = \delta^0$ is the dimension of a single point, or any finite sum of single points. This dimension structure results in particular when $\mathbb{D}$ is the category of bounded polyhedra, but on the other hand, if $\mathbb{D}$ includes both bounded and unbounded polyhedra, Schanuel showed that the dimension Rig itself is multidimensional. But a further remarkable result of Schanuel is that in both those examples the two reductions of the Burnside Rig are jointly faithful; that is, an object is determined up to isomorphism if we know both its Euler characteristic and its dimension. That result has the same flavor as the famous Poincaré conjecture, which would have it that certain spaces are determined if we know both their homotopy type and their dimension. Note that the homotopy type determines the Euler characteristic, thus one might speculate that if the three-sphere were considered as an object in the appropriate distributive category, there might be a whole cloud of refined dimensions corresponding to coarse dimension three and that Poincaré might at least be right in this weakened sense.

The significance of such quantities will depend on whether $\mathbb{D}$ is a general category or a particular category; with this I refer to the following distinction:

The category of all spaces of a certain general kind is distributive, but if we consider a particular space and look, for example, at the category of set-valued sheaves on that space, the latter will also be a distributive category, giving a kind of glorified picture of the space in terms of the totality of continuously variable cardinals that it can parameterize. Thus both, the general and the particular, (although qualitatively different, as I will show) do share the property of distributivity. And this is why distributive categories are offered as a general model of space both in the general and particular sense.

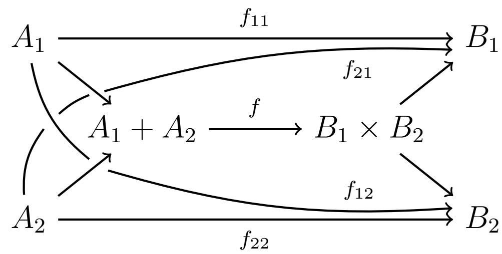
Quantity

Grassmann says that quantity is that which increases and that numbers arise when we consider the difference, that is, a certain quantity increases from A to B then there is a corresponding number $B - A$. Thus, points in space could be quantity with vectors the corresponding numbers, or energy a quantity with work the corresponding kind of number. In this terminology there is no quantity "zero", but there is a number 0, since it might happen that in going from A to B a particular quantity does not actually change. Thus, the categories to which I will refer as quantitative, would perhaps better be called numerical, except lately "numerical" tends to have a more narrow meaning suggesting perhaps discreteness. I will call these categories linear categories; though categorists for the past 30 years have tended to call them additive categories, I feel that the term linear will be more directly suggestive to physicists.

The initial object 0 has by definition a unique map to any object and from any object there is a unique map to the terminal object 1; thus, in particular, for either, or both of these reasons, there is a unique map from the initial object to the terminal object in any category where both of these exist. In a nontrivial distributive category this map is never isomorphic; it expresses the inclusion of the empty set in the singleton set. By contrast, the first axiom for **linear** categories is that this map **is** an isomorphism, so that the two kinds of objects coalesce into one, commonly called a zero-object. Then for any two objects A and B there is a distinguished constant map called the zero map from A to B, constructed as the composite

$$A \longrightarrow 1 \xleftarrow{\ \sim\ } 0 \longrightarrow B \qquad 0_{AB} : A \to B$$

Now, in any category which has coproducts and products, an arbitrary map from a coproduct to a product is uniquely determined by a matrix of component maps.



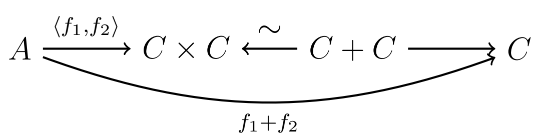
The components of f are determined by composing with the coproduct injections and the product projections and conversely, the matrix of these components may be considered as a name for f itself

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}.$$

In particular, in a linear category there is a canonical map from the coproduct of any objects to the product of the same two objects, namely, the one whose matrix is the "identity matrix", i.e. the one which has the identity maps of the two objects as diagonal components and the two zero maps as the off-diagonal components

$$A + B \xrightarrow{\begin{pmatrix} 1_A & 0_{BA} \\ 0_{AB} & 1_B \end{pmatrix}} A + B$$

The second axiom for linear categories is that this identity matrix is an isomorphism. This has the consequence that maps can be added for, given any two maps from $A \to C$, we can define their sum as the composite



Here the first paired map is the one guaranteed by universal property of products, the third is the co-diagonal (co-pair of 1_C with itself), which exists in any category with coproducts, but the middle is the inverse of the identity matrix. In a linear category the coproduct (equals product) functor is often denoted by $\bigoplus$ to distinguish it from the addition of maps. Note that the addition of maps is intrinsically determined by the category structure; because it is intimately related to the product and coproduct functors which are naturally commutative and associative, it can be shown that the addition of maps is also commutative and associative, and moreover has 0_{AB} as neutral element; moreover, the composition in the category by maps $A' \to A$ or $C \to C'$ distributes across the addition of maps. In a linear category it is in the first instance the maps themselves that are to be considered as quantities (more properly numbers) whereas any pair of objects determines a "type" of quantity. In particular, if we have an object A whose endomaps happen to all commute, then the quantities of type $A \to A$ automatically form a Rig; note that non-commutative quantities typically occur in contexts where there are several types.

Does this give us enough space to deal with intensive/extensive dimensional analysis, vectors versus tensors, constant versus variable? Yes, provided we consider the relationships between various linear categories and various distributive categories inside the context of a particular or fixed notion of the distinction between the general and the particular. From the quantitative point of view "general" means that "quantity is space" so that the idea of space, namely that one can move, cohere and vary can be applied within the realm of quantities of a given type. That is, the set of quantities of type $A \to B$ in a certain linear category $\mathbb{L}$ may be more than just an abstract set in that it is actually the set of points of an object $\mathbb{L}(A, B) \in \mathbb{D}$ in a certain distributive category $\mathbb{D}$. If $\mathbb{D}$ is a **general** distributive category, then this relationship is the crucial basis for the subject known as Functional Analysis. On the other hand, if we have a given differentiable variety, then the **particular** distributive category of sheaves on this variety also enriches linear categories such as the category of vector bundles on the variety which is to be construed as a particular linear category. How, more concretely, do space and quantity interact? From the point of view of the quantitician spaces have no other role than to act as domain of variation for variable quantities. From the point of view of those who want to get somewhere, spaces have the role of the arena in which motion can take place. If X is an object in a distributive category $\mathbb{D}$, then X can be the domain of variation of many variable quantities, but the very distinction between space and quantity reflects for particular X into a distinction between two broad classes of variable quantities: the extensive quantities which somehow measure space directly and the intensive quantities, which measure them as quantities, intensively. We want to turn this philosophical distinction into mathematics and fortunately we can make some initial step in that direction. In general, a covariant functor $\mathbb{D} \xrightarrow{E} \mathbb{L}$ could be called a type of extensive quantity if

$$E(X_1 + X_1) \cong E(X_1) \times E(X_2)$$
$$E(\varnothing) \cong 1 \ (= \{0\}),$$

with $E(X)$ thought of as the linear space of extensive quantities of type E varying over X. And in particular $E(1)$ is the space of constant quantities of type E. Notice that if we have any given point $1 \xrightarrow{x} X$ of any given object X in $\mathbb{D}$ we can apply the covariant functioriality of E to obtain $E(x)$ which is a linear map from $E(1) \to E(X)$ that should be considered as the Dirac δ_x an extensive quantity concentrated at the point x of X (more exactly, $E(x)$ maps any constant quantity λ of type E to the multiple $\lambda\delta_x$. A sum of several such is a typical Riemann sum, also an extensive quantity of type E on X. Thus, the Dirac δ, far from being a mysterious non-function, is actually the simplest case of an anti-function, i.e. an extensive quantity rather than an intensive one; the whole mystery about it was created by the idea that it should be like a function, that is to say an intensive quantity, which it is not. Another concept that we can derive in this generality is the notion of the total value of an extensive quantity, namely, if we apply the functor E to the unique map from $X \to 1$ we get a well-defined linear map $E(X) \to E(1)$ which assigns to any extensive quantity of type E varying over X its total value, which is a constant of type E, for example, the amount of cigarette smoke in this room. Another example of the functoriality with neither domain nor codomain $= 1$ is to consider a path $T \to X$ where T is (as spatialisation of) a time interval. A standard extensive quantity on T itself would be a duration, an element of $E(T)$ applying the functor to the path and evaluating it at duration, we obtain an extensive quantity of the duration type E but varying over the space X; this extensive quantity on space is called the sojourn of the particular path. The sojourn of the path is still a variable extensive quantity since I am for a certain duration in Cambridge, for a certain duration on the motor way, etc. Basically Grassmann justifies his book "Ausdehnungslehre" with the observation that the extensive quantities are widely neglected. This is still true, and indeed there is a kind of conspiracy against extensive quantities or a reluctance to consider them as such, as exemplified by the attempt to call Schwarz distributions generalized "functions". Distributions of compact support are probably the most basic example E of

Extensive Quantities on the particular distributive category $\mathbb{D}$ of C^∞ spaces. Similarly, there is the idea that measure theory is difficult and that it would be simplified by replacing it with the study of density with respect to a fixed measure such as length or volume. That to a certain extent this is possible is due to the relationship between extensive and intensive quantities which can also be explained in general terms. An intensive quantity type C differs from an extensive quantity type in at least two respects. First, it is contravariant, rather than covariant – as a functor, and second, it has not only a linear structure, but also a multiplication. For example, if R is a fixed Rig in $\mathbb{D}$ it could define a contravariant functor C by $C(X) = \mathbb{D}(X, R)$ so that in particular $C(1)$ equals points of R equals constant intensive quantities of type C. If $\mathbb{A}$ denotes the category of all abstract Rigs, $\mathbb{L}$ the category of abstract commutative monoids, $\mathbb{L}_R$ the category of abstract R-linear spaces, we obtain a commutative square of functors as follows:

$$
\begin{array}{ccccc}
\mathbb{D}^{\mathrm{op}} & \xrightarrow{C} & \mathbb{A} & & \\
 & \searrow & & \searrow & \\
 & & \mathbb{L}_{\mathbb{D}(1,R)} & \longrightarrow & \mathbb{L}
\end{array}
$$

where $\mathbb{A}$ is the category of abstract Rigs and $\mathbb{L}_{\mathbb{D}(1,R)}$ is as stated above. Of course, to take advantage of functional analysis, we would consider an $\mathbb{L}$ enriched in $\mathbb{D}$. The contravariant functoriality of this kind of intensive quantity is just given by composition in $\mathbb{D}$. Such "substitution" of course preserves addition and multiplication and so gives rise to a morphism in $\mathbb{A}$. By appropriate choice of $\mathbb{D}$ and R these quantities may turn out to be differential forms, rather than continuous functions. But there is also a relation between intensive and extensive quantities, for if α is an extensive quantity on X, and f an intensive one of the appropriate type on the same X, then there will be defined another extensive quantity $f \cdot \alpha$ on the same X. For example, mass = density·volume often holds. For a particular mass distribution such an equation might also **not** hold for any density function; that is, division is not always defined, but the multiplication always is. As a special example E might be defined as the linear dual of $C(X)$

$$E(X) = \mathbb{L}(R^X, R).$$

In this case the action of intensives and extensives can be explicitly defined by requiring that for all g

$$\langle g, f \cdot \alpha \rangle = \langle gf, \alpha \rangle.$$

There is an important identity which links together the covariant functoriality of the extensive quantity, the intensive functoriality of the contravariant quantity and the action of the latter on the former. This identity can be directly verified in the example just given, but should be required even in more complicated cases. The identity may be described in words by saying that if we consider an arbitrary map ϕ from a space X to a space Y, then $E(\phi)$ is linear even with respect to the variable scalars from the Rig $C(Y)$. Here, the action of the latter variable scalars on $E(X)$ is defined by combining $C(\phi)$ with the action at X. This formula appears in all parts of mathematics under all sorts of different names. In Quantum Mechanics it is called the canonical commutation relation (when taken in differentiated form) in algebraic topology it is called the projection formula, and in the theory of group representations it is called Frobenius reciprocity. In the latter case, both the intensive and extensive quantities are constructed in the objective Burnside/Steiner manner. The identity even occurs in logic in the form of a certain commutation between existential quantification and conjunction. For an example in logic in which extensive quantities are not the linear dual of the intensives, we could construe intensive quantity to mean propositional function and extensive to mean a finite subset. The latter is not a contravariant functor, but is covariant and admits the required action by propositonal function, (with the contravariance given by substitution of any map ϕ into a propositional function on the codomain of ϕ.) And the covariant functoriality is given by the restriction to finite sets of the existential quantification or direct image which is left-adjoint to substitution. In the above, we were thinking of a general category of spaces, but the discussion also applies to particular distributive categories associated with a particular space, such as the sheaf category.

The quantities actually become rather different in the particular context. Maps between vector bundles become arbitrary differential operators. The tangent bundle becomes a Lie algebra, which it was not in the category of all smooth spaces.

Now I'd like to say something more about logic: that it does not just arise spontaneously, but is actually more organically connected to the more general quantities. Namely, given a variable quantity, we can ask "where is it?" "Where does it live?" The general quantity varying over a space X has many values, but we can ask "where is it zero, and where is it not zero?" A "two-valued" or "logical" question! The idea of the set of zeroes as an intensive quantity was emphasized by Descartes in his well-known "Analytic Geometry". On the other hand, extensive quantities don't have values at points, so to raise the question "where do they live?" is to ask for their **support**. The support of a general extensive quantity ought to also be an extensive quantity, but of a logical or idempotent kind, something like the above example of finite sub-sets. Although the concept of support is frequently used, in particular applications, I have never seen a general philosophical or logical account. We can define an ideal which should be principal in the case, when the support in a narrow sense exists. Namely, when is an extensive quantity $\beta \in E(X)$ supported on a subjobject $A \overset{i}{\hookrightarrow} X$? (we might restrict ourselves to closed subspaces **that is**, those definable as equalizers of intensive quantities on X, especially when the extensive quantities are identifiable with functionals on the intensive quantities): If there exists a member α of $E(A)$ such that $E(i)(\alpha) = \beta$. **The** support ought to be the smallest such A. However, sometimes the intersection of all the A on which β is supported will no longer support β.

The idea of supports of a kind of extensive quantity would seem to lead to an extensive logic of idempotent quantities, just as important as the usual intensive logic of propositional functions. With its covariant functoriality it usually leads to different lattices than the corresponding intensive logic. This is a program: To develop the **logic** of interrelated intensive and extensive quantities **as a particular kind of simplified view** of the more general quantities; understanding all about where the quantities live, is a first approximation to investigating the quantities themselves.

There is another program, another kind of simplified view, which is homotopy theory. Recall the Cantorian analysis of a distributive category $\mathbb{D}$ in terms of the dialectics of a lower Boolean category of Kardinals. Often there is a further feature, namely a further left adjoint π_0 (such a π_0 functor usually does not exist in the higher stages of the dialectical reconstruction of $\mathbb{D}$ from $\mathcal{K}$), which enables us to restate the condition that a space X is connected iff every map from X to a discrete space is constant by saying simply that $\pi_0(X) = 1$. If $\mathbb{D}$ happens to be general (but definitely **not** when it is particular) this components functor may have the two further properties that it preserves finite products and in case $\mathbb{D}$ is actually a topos, that $\pi_0(\Omega) = 1$. For example, this happens when $\mathbb{D}$ is the category of reflexive graphs which we discussed before. Let us concentrate on the product preserving property

$$\pi_0(X \times Y) \cong \pi_0(X) \times \pi_0(Y).$$

It is not true for G-sets with G a group; this is really a particular topos as Hopf in effect knew already in the 1940s. It holds for M-sets for M a monoid iff M has at least one constant. An important consequence of this product preserving property when $\mathbb{D}$ is cartesian closed, is that we can define a second category $\pi\mathbb{D}$ with the same objects as $\mathbb{D}$, but with Kardinals of maps given by the Kardinal of components of the functions space

$$\pi\mathbb{D}(X, Y) = \pi_0(Y^X)$$

instead of the original $\mathbb{D}(X, Y) = \text{points}(Y^X)$. This gives a category only because the product-preserving property enables us to define composition of these components. Namely, we just apply the π_0 functor to the internal composition map in $\mathbb{D}$ itself. The intuition behind the homotopy category is that if one map can be continuously deformed into another map, then the two are regarded as already equal. The components functor lifts to a functor $\mathbb{D} \to \pi\mathbb{D}$ which assigns to any space its "homotopy type". A category $\pi\mathbb{D}$ is again a distributive category, even cartesian-closed; however, it usually does not have equalizers. The qualitative shape of a space X is thus interpreted also as an object of a category, drastically simpler than X itself. This is another way of getting a partial picture. Instead of asking, as logic does, "Where is it?" or as the general quantities do "How much?" we look at the qualitative shape, which is itself a kind of space since it is again an object of a distributive category. Are there any intensive and extensive quantities to find on these "spaces" in $\pi\mathbb{D}$? One could approach this in the concrete manner by considering a particular topos $\mathcal{P}(X) \hookrightarrow \mathbb{D}/X$ and considering the Burnside Rig of **its** homotopy category, or follow a more algebraic road as is usually done; but the basic result is cohomology as intensive and homology as extensive quantity types defined on $\pi\mathbb{D}$.

Cohomology classes act on homology classes in a way that satisfies the projection formula. Can this be an accident? **Wesely Phoa**: Yes! (laughter). Recall that the central example of intensive quantity was simply R-valued functions, i.e. that they form a representable functor. This feature also persists into the qualitative domain, according to an important 1943 theorem of Eilenberg and Mac Lane which states that for each Ring R in $\mathbb{D}$ there exists a sequence of spaces called $K(R, n)$ so that

$$H^n(X, R) = \pi\mathbb{D}(X, K(R, n)).$$

This is a fitting equation with which to end a lecture wholly devoted to philosophy.

Bell: Probably you have exemplified in the lecture for one advertised topic, but you haven't commented on explicitly "the need for objective logic, as well as subjective".

Lawvere: It was, I think, the position of Aristotle and Hegel that logic is the science of the development of thinking. So that, in particular, there is the logic of the building up of concepts which is the subject of the Aristotelian categories and Hegel's process of building up by the unified process of "Aufhebung", the categories of Being, Becoming, etc. These are laws which describe the everyday development of scientific thinking, laws for constructing the mental objects about which statements are made. Important in mathematics, but hardly any other field, are the precise laws for the deduction of one judgement from another. The laws of the process of deduction are descriptive, not of mental objects reflecting reality, but of a more purely mental process, and for that reason are called subjective. On the other hand, the part of logic which describes the struggle to construct these mental images of Being, is called objective logic.

It is a shame that objective logic has been so neglected. Of course Aristotle and Hegel made great contributions to it, but the 150 years of further experience of the development of science which we have had should make it possible to go much further and produce a usable tool for the guidance of science.

There seems to be an implicit idea nowadays that objective logic might be reducible to subjective logic, along the lines if I start talking, for example, about space and quantity, then you can say "define what you mean" - then you are making just statements from which other statements can be deduced.

How did I arrive at these ideas? In fact, Grassmann had the opposite reduction in mind, and indeed I showed 25 years ago that all the rules of inference of subjective logic can be seen as special cases of the objective principle of adjointness (Dialectica 1968). ((The lectures I gave in Los Angeles 1967 and remarks made in Berkeley in 1963.)) But of course, still more important is the dialectical relations between the two.

The additivity property in the general definition of an extensive quantity type can often be strengthened through detailed study of particular examples to give some general rules of calculating the extensive quantities on particular pushouts, that is, spaces obtained by glueing together two other spaces, but with a controlled overlap which is non-empty.

Response to a question of **Dorling**:

Lawvere: The idea in physics that a group action on input induces a group action on output is a special case of functoriality and of course occurs in both covariant and contravariant, as well as in general and particular contexts. I haven't yet understood the full connections with so-called quantum groups.

Answer to **Auditor**:

Lawvere: There is a recent book by Frölicher and Kriegl that deals with the structure involving a countable set of norms.

Question from audience: Do you think it advisable to equate subjective vs. objective with intensive vs. extensive.

Lawvere: This is a serious question which I am perplexed about myself; there is obviously a very strong analogy. Perhaps someone more versed than I in Philosophy could show me a precise derivation of one from the other.

Lengthy enthusiastic applause.

Acknowledgements:

We are happy and grateful that Francisco Marmolejo, with his many skills, his insights and serious dedication to mathematics, reformatted the Lecture Series for the Posthumous Publications.

Michael Wright conceived and organized the unique events of June 18 - 25, 1989; unique at that time because, both practicing mathematicians and philosophers gave talks and discussed. We thank Michael for this initiative and for his ongoing archival work and collaboration.

Bill's three-part lecture series was the Invited Address, which Michael Redhead chaired in the Old Cavendish Laboratory. This series preceded the "Workshop on Category Theory and the Foundations of Mathematics," with talks by John Bell, Anders Kock, John Mayberry, Colin McLarty, Alberto Peruzzi, Gonzalo Reyes, et al. Redhead was the host for the Workshop at his own Wolfson College, Cambridge, and his wife Jennifer Redhead hosted the social gatherings. Thanks to all.

Bill was happy about the conference, and grateful for the reception and hospitality. He enjoyed that there was a historical link with James Clerk Maxwell (one of his heroes), the developer of electromagnetic theory, who had been the founder of the laboratory, and the first Cavendish Professor of Physics.

Part of the transcript of the first lecture became the basis for the publication: Cohesive Toposes and Cantor's 'lauter Einsen' in Philosophia Mathematica, vol. 2, 1994. Bill was very grateful for the transcription work from audio by John Bell and Michael Wright, and so are we!

The Archive Family