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A Philosophical Guide to Mathematics based on the Objective Dialectics of Category Theory

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I

Category Theory in the Foundations of Mathematics

June 14, 1989

Why must we study the foundations of Mathematics?

Why must we study the history of Mathematics?

According to my interpretation we must study the history of Mathematics in order to arrive at the foundations of mathematics, in order to discover the laws of the development of scientific thought, objective as well as subjective.

Why must we know these laws?

Above all to arrive at the position from which we can teach these laws.

And why teach them?

In order to provide an explicit assistance to the *Learning*, *Development* and *Utilisation* of mathematics itself, that is, the science of space and quantity.

There is however another conception, according to which one studies the history of mathematics in order to demonstrate, let us say, God talked with Plato and Plato somehow talked to von Neumann and the great mass of mathematicians (those who actually develop and utilize mathematics) have nothing else to do but to deduce consequences from the axioms received in that way from von Neumann, and in the last analysis from God.

It seems that every true advance in the search for the laws of the development of the laws of mathematical thought gives a great contribution on the one hand, but on the other hand can be exploited by the second-mentioned line in the following way:

There is first a period of open resistance against the new development, and against its content which start to make clear that everyone *could* participate in the developments and utilisations, hence in particular, to learn and not to remain a slave who can only do certain calculations and mechanically deduce certain consequences from axioms which arrived from "somewhere else".

After a period of that kind of open resistance it becomes too difficult to maintain that point of view and one passes from rightist positions to an "ultra-left" position. That "ultra-left" position has at bottom the same purpose: they pretend to accept the new results and they say: "but why don't we accept even more; we can do everything, right away!" which means that instead of studying the results of the first line, in a serious way (which of course would involve effort), one should introduce a completely "new" and "revolutionary" system, as Dewey, for example, in 1919 in China preached to the Chinese teachers that each student should develop "his own" system, that one should not teach new science, i.e. the new valid knowledge which was found by means of investigation and study, instead that it would be "more important" that every child should develop his own system without study, because to force him to study would be authoritarian. In this sense the "ultra-left" line of Dewey is in substance a way to promote the avoidance of study, even though it pretends to be an acceptance of the new and recent results. In the end the desired result is that the students see that these “revolutionary” systems developed without serious study do not serve and hence from the resulting insecurity, return to acceptance of the old rightist line. This type of movement one also finds in the history of modern mathematics.

Five years ago I wanted to give a course in Set Theory. The set theorists were strongly opposed, however the greatest among them, that is John Myhill, instead supported my attempt. On the first day of the course I tried to explain the invariant content of the category of sets of which I had an axiomatic description, but wanted to give an ideological vision of what could be the significance of the objects of this category - nothing more or less. So I explained, using the expression “abstract set” - a set which contains certain points which are deprived of content, and a set itself deprived of any structure except the fact that two points may be equal or unequal - . Myhill (who unfortunately is now dead) came to the first days of the course, heard this description and said to me: “I heard this somewhere before”. “Where?” “In Cantor”. He brought me his copy of Cantor’s works with a note saying: “see page 283 where Cantor speaks of “...lauter Einsen”. (This is a somewhat strange expression in German, because “eins” means “one” yet it is pluralized and nominalized.) In fact, studying these pages, one sees that the set-theorists of this century have forgotten an important component of what Cantor himself rather clearly explained about the basic conceptions of set theory. Cantor speaks on the one hand of “Mengen” and on the other hand of “Kardinalen”. “Mengen” is normally translated as “sets”, and in every book on set-theory they speak of “cardinals”, but their “cardinals” are completely different from those described by Cantor himself.

> How did things arrive to this point?
>
> Perhaps it's not important?

One would think, in studying the history of set-theory, that it would be important; it would seem that in a book dedicated to the study of Cantor and to his set-theory. (In a recent book written by a student of John Bell in England, entitled: *Cantor, the theory of sets and the limitation of size*, one does not speak of Cantor's "Mengen", nor even of "Cardinals" in the sense of Cantor.)

Cantor himself, in an article summarizing his development of set-theory, speaks of "Mengen" and of the passage from them to "Kardinalen". Myhill found that this description of "Kardinalen" by Cantor and my description of abstract sets were essentially the same. "Mengen" as such are not treated explicitly in any way in the books on set-theory of this century. They must, of course, in some way be treated in order to pretend to have any relation at all with mathematics, but as theoretical objects in the manner of Cantor they are not dealt with.

An excuse for this omission could be, as we have already found in other examples of the works of great mathematicians of the last century, that the editors themselves of the collected works of great men, have a great influence in the sense that any scholar desirous of understanding the work of the great one, knowing that it is difficult, naturally goes to read the comments of these editors "who must know". The need for a guide always exists and it is they who have provided us with this guidance. The editor of the works of Cantor writes that that part is inconsistent, even if Cantor is great, and it cannot be made consistent and that in any case it is more important to pass immediately to the arithmetic of cardinals, cardinals in the sense of this century. More precisely, the editor says that this attempt to explain cardinals as the result of a process of abstraction involving the "lauter Einsen" (which I will return to explain more fully in a moment) was "not a happy one", because these "Einsen" elements must be different from one another, but how can they be different if they have no distinguishing properties? This contradiction, which is really a contradiction in a productive sense as I will demonstrate, led the editor to say that the whole concept is inconsistent and hence that one cannot speak of cardinality in this manner and, therefore, that it would not be possible to move forward to interesting cardinality calculations even though Cantor himself did exactly that.

Thus, it seems, one must begin again from the beginning. One must study all the mathematicians of the past century, work and study to find the core content, without being prejudiced by the common opinions of the editors of collected works and others during the period after 1894.

One knows that the concept of equicardinality of "Mengen" is somehow concerned with a kind of isomorphism which in Cantor is called "Mächtigkeit", that is, two "Mengen" are equally potent if there exists a bijection between them. But a very interesting point which emerges from reading the work of Cantor is that he himself cites the origin of this word "Mächtigkeit" in the work of the famous Swiss geometer Jakob Steiner, who apparently used this word to signify isomorphism in a *different* category, in fact the category of algebraic spaces, in his work on conic sections (1850), where equivalent potency was used to explain that the ellipse is not equivalent to the parabola, nor the parabola to the hyperbola, in an intrinsic geometrical matter. These are all objects in one simple category in which there are many different ellipses, many different hyperbolas, but there is a concept of isomorphism of which the usual invariants exemplify invariants of isomorphisms. According to Cantor himself, he took this concept of isomorphism from this geometric context and arrived at *his* concept of isomorphism, naturally in a more abstract context. Also this fact is not to be found in any book that I have seen (although it is emphasized in a recent paper of Colin McLarty who noticed it independently from me). Naturally, here one can say that in embryonic form one finds already the concept of category, because in the analogy between algebraic spaces and abstract sets there is already implicitly a unifying concept. Cantor himself says that his concept of Mächtigkeit is **different yet similar** to that of Steiner. Therefore one can speculate that if the set-theorists had only studied Cantor with more seriousness, they would have discovered the theory of categories 50 years before Eilenberg and Mac Lane. But the "foundational" culture has somehow blocked even the great Swiss set-theorists who are also fond supporters of Steiner, from noticing the significance of this connection.

The third example of a mysterious fact we noticed in reading the works of Cantor is **Meta-mathematics.**

What is Meta-mathematics?

The common opinion, and this common opinion has been supported by Kleene's book which is precisely about meta-mathematics and by various encyclopedia articles, is that Hilbert somehow started to talk of meta-mathematics around 1900. However, already in 1883 Cantor spoke of meta-mathematics and indeed not in a favorable sense. He speaks instead of a particular anti-mathematical current of positivists which sought to put in doubt the general validity of mathematical proofs and mathematical truth. Unfortunately he does not specifically name the persons involved. But he does refer to this as a group of sophists who are attempting to destroy the general faith in mathematics and says that they use the slogan "meta-mathematics" themselves to describe their tendency. One can speculate as a problem for further detailed historical research that Cantor's own later retreat into the madness of objective idealism was in part a reaction to this subjective idealist attack on his beloved science.

Let us now return to a more precise description of what are these "Mengen" and "Kardinalen":

The citation of Steiner demonstrates to me that Cantor must have studied the work of Steiner, for I myself had to read many pages of geometry in Steiner's book before I could find the word "Mächtigkeit". We know, in fact, that Cantor worked first in the theory of numbers, then on the theory of Fourier series, particularly with respect to the sets of convergence and of uniqueness of such series and then arrived at the necessity to study sets in a general manner from these examples and then, finally, at the necessity to study the cardinality of these sets as a kind of first invariant of a problem.

Thus a “Menge” has indeed an ensemble of points, but more than that, also is *cohesive* and *variable*, a feature which the abstract sets do not have, except in a degenerate way. Today one habitually speaks of “topology”, for example to indicate a cohesive structure which a line has (a line on which might be defined a function which might have a Fourier series, etc). However, we think that, yes, there is a topology in the usual sense, but also many other similar structures which might be better adapted to particular problems. One must study not only these particular technically defined structures, but one must also have a general conception of spaces with cohesion. I think that Cantor probably also thought in this way. Without entering into the acceptance of a particular definition of what cohesion might consist of, we can accept in general the existence of spaces with cohesion. Naturally to make a detailed analysis, we need eventually to find appropriate, detailed definitions, but for the purpose of a general analysis one can also treat these objects in a general way.

This cohesion of a topological type we may refer to as an *objective* cohesion; on the other hand there is also cohesion of a *subjective* type, i.e. we come to know these points in a certain way (perhaps they are the values of a particular recursive function), that is, in my subjectivity, in my knowledge of a certain “Menge”, I may come to know a certain point before I come to know another one. The succession of the appearance of the points usually does not have an objective mathematical significance, but nonetheless may have an interest in certain connections, having to do, for example, with methods of calculation. This too is a type of cohesiveness. Thus a recursive set is traced by various threads given by particular recursive functions and this too gives it a kind of cohesiveness which by contrast with the other we might call subjective. Both of these types of cohesiveness were recognized by Cantor. This is why Cantor used a double bar to indicate the double abstraction which in general may be involved in passing from a cohesive Menge to its associated Kardinale.

Let us suppose that $\mathcal{M}$ is a particular category of Mengen, for example the category of topological spaces in one sense, the category of topological spaces in another sense, the category of recursive spaces, one or another category of combinatorial spaces, etc. Cantor says that we can take the pure set of points of any such space, thus arriving at a cardinal. A contemporary illustration of this process might be to depart from a color television picture with subtle contrast of color and detail provided by highly advanced technology. But then we can turn down the color knob and turn up the contrast knob until nothing remains but stark white dots on a black background with even, we may imagine, the outlines of figures suppressed. The picture with all its beautiful colors is a "Menge"; but in order to concentrate on the study of a certain superficial (but necessary) aspect, one may consider that bag of points obtained in this way. With this process of abstraction one forgets temporarily all the beautiful particularities, in order to concentrate only on the points now deprived of qualities, but still equal in their numerosity to those of the colored picture. This seems to be also a Menge, but a Menge of a degenerate sort. That is, we may consider that every cardinal gives rise to a Menge of a type called discrete and that in fact we have a pair of adjoint functors

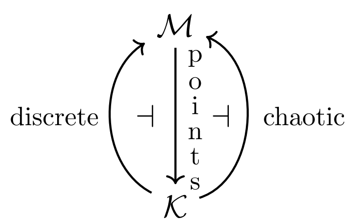
$$\begin{array}{c} \mathcal{M} \\ \text{discrete} \left(\ \dashv \ \right\downarrow \text{points} \\ \mathcal{K} \end{array} \qquad\qquad \begin{array}{c} \dfrac{K \longrightarrow \text{points}(M)}{\text{discrete}(K) \longrightarrow M} \\[1em] \text{discrete}(\text{points}(M)) \longrightarrow M \end{array}$$

Thus a map of abstract sets from a cardinal K to the points of a Menge M is a map of abstract sets, that is, a map of a completely general kind with no condition of "continuity" or preservation of cohesiveness, but to give such, is equivalent to giving a continuous map from discrete(K) to M. The continuous maps in the other direction are by no means arbitrary for most M; for example, there will be no non-constant $\mathcal{M}$-maps $M \to$ discrete(2) if M is *connected*; in fact, for many $\mathcal{M}$ the foregoing clause serves well as a definition of which objects M of $\mathcal{M}$ are to be considered as connected,[1] for the morphisms of $\mathcal{M}$ must preserve the cohesion of which M may have much but of which discrete(2) has none; thus a non-constant map of the kind indicated is only possible if there is a break in the cohesiveness of M. As usual with adjoints, taking the case where K is points(M) and taking the identity map in K we obtain a canonical map in $\mathcal{M}$ from discrete(points(M)) to M itself which may be considered as the best approximation we can make

[1] This definition of connectedness is correct not only because it leads to useful technical consequences, but more basically, because it corresponds to an objective concept: a "Menge" has cohesiveness in general, but might in particular have two parts, each cohesive in itself, but with no cohesiveness "between" the two parts in question; hence an $\mathcal{M}$-map would exist preserving all the coherence that there is to preserve, but mapping the two parts to the respective points of the space discrete(2) whose cohesion is nil.

"from the left" to M knowing only its cardinality.

But the points functor has also a *right* adjoint, which is sometimes called codiscrete or chaotic. The chaotic space determined by a cardinal is usually completely different from the discrete space determined by the same cardinal, so that, for example, all $\mathcal{M}$-maps chaotic$(K_1) \to$ discrete(K_2) are *constant*, whereas the $\mathcal{M}$-maps in the other direction are (for two reasons) as many as the arbitrary $\mathcal{K}$-maps $K_2 \to K_1$

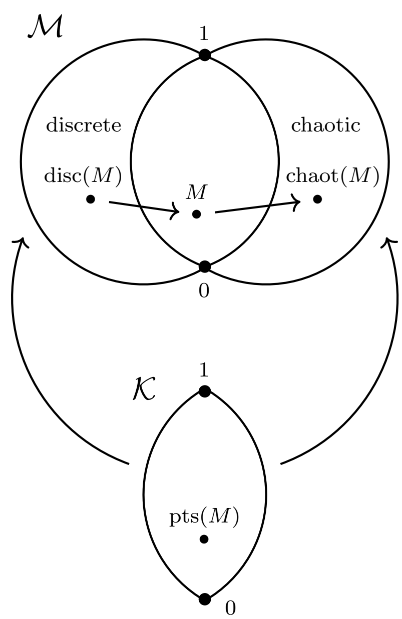


$$\frac{M \longrightarrow \text{chaotic}(K)}{\text{points}(M) \longrightarrow K.}$$

Here again the horizontal bar is an abbreviation to indicate that there is given a natural bijection between the $\mathcal{M}$-maps with domain and codomain as indicated above the bar and the $\mathcal{K}$-maps as indicated below the bar.

In general a discrete space is completely deficient in its cohesiveness so that each point remains forever itself and no motion is possible, i.e. no map from a connected space to it can pass through two distinct points, whereas by contrast a chaotic space is so excessive in its cohesiveness that any point can be moved to any other point without any "effort", i.e. without any regard to the nature of the space-"time" which might parameterize such motion. Also this excessive cohesion which a chaotic space has, is completely determined by its cardinality. Usually the chaotic spaces are in a trivial way connected, both because one can pass continuously (i.e. by a notion parameterized by any connected parameterizer one wishes, provided it has at least two points) between any two points, as well as because the maps from it to a discrete space are all constant as in the formal definition offered above.

There are in fact interesting categories of Mengen (combinatorial and/or bornological in nature) which are in a certain sense[2] generated by the chaotic objects only, even though having objects with arbitrarily complicated higher connectivity properties. However, these are extreme special cases and we



The two composite functors are both isomorphic to the identity of $\mathcal{K}$:

$$\text{discrete}\cdot\text{points} \cong 1_{\mathcal{K}} \cong \text{chaotic}\cdot\text{points}.$$

want to continue for a while to discuss the more general situation in which

[2]That is, for such a special $\mathcal{M}$, knowing all about the special maps chaotic$(K) \to M$ completely determines the arbitrary object M - such have been studied for over 50 years by topologists under the name “Simplicial Complexes”.

we consider a very arbitrary category of Mengen, in which there are the two opposed sub-categories of discrete and codiscrete objects, each in itself identical with a category of Kardinalen, that is, if to the canonical map $\mathrm{disc}(\mathrm{pts}(M)) \dashrightarrow M$ we apply the functor "points", we obtain an isomorphism of cardinals

$$\mathrm{points}(\mathrm{discrete}(\mathrm{points}(M))) \overset{\sim}{\dashrightarrow} \mathrm{points}(M),$$

and similarly with the canonical map

$$M \dashrightarrow \mathrm{chaotic}(\mathrm{points}(M))$$

applying the functor "points" we obtain an isomorphism of cardinals

$$\mathrm{points}(M) \overset{\sim}{\dashrightarrow} \mathrm{points}(\mathrm{chaotic}(\mathrm{points}(M)))$$

even though the two original canonical maps themselves are usually very far from being isomorphisms of Mengen. Thus the contradiction objectified in the system of adjoint functors explains the "inconsistency" which blocked the progress of Zermelo (Cantor's editor) from understanding this foundational question, for indeed the points of a cardinal are definitely distinct as is shown by how the discrete spaces relate to all the other spaces in $\mathcal{M}$. On the other hand its points are completely indistinguishable as is shown by the manner in which the corresponding chaotic space behaves in relation to $\mathcal{M}$.

The "inconsistency" of diversity versus indistinguishability, of having a definite number of points, but of these points being indistinguishable by any property, seem to Zermelo so antagonistic a contradiction that nothing coherent could be done. But the explicit use of adjoint functors between categories in this configuration lays everything out so that the productive nature of the contradiction can become clear to everyone. The adjunction map

$$\text{discrete}(\text{points}(M)) \dashrightarrow M$$

is monomorphic so that the points remain distinct as they were in M itself. On the other hand, in chaotic(points(M)) not only are the points "equal" in the philosophical sense that there are no distinguishing properties, but even in the stronger sense that one can pass from one to the other along motions or variations parameterized by any desired connected space.

Unfortunately, we have been blocked for almost a century from understanding clearly and developing further these simple but powerful modes of thought, because most of the set theorists have followed the guide as exemplified by the editor of the collected works of Cantor. The Frege ... von Neuman concept of "cardinal" is too abstract because it can no longer carry the all-important notion of map and hence does not give rise to a category $\mathcal{K}$.

$$\text{discrete}\cdot\text{points} \cong 1_{\mathcal{K}} \cong \text{chaotic}\cdot\text{points}$$

$$\frac{\text{discrete}(\text{points}(M)) \xrightarrow{\text{canon}} M}{\text{points}(\text{discrete}(\text{points}(M))) \xrightarrow{\sim} \text{points}(M)}$$

$$\frac{M \xrightarrow{\text{canon}} \text{chaotic}(\text{points}(M))}{\text{points}(M) \xrightarrow{\sim} \text{points}(\text{chaotic}(\text{points}((M))))}$$

The question from the audience from a geometer who objects that he does not feel he needs to know the axioms of set theory because all these fine distinctions made by the foundationalists do not seem to have any bearing on his geometry. The speaker responds: "but that is exactly what I am trying to say!"

What are these so-called foundations of mathematics?

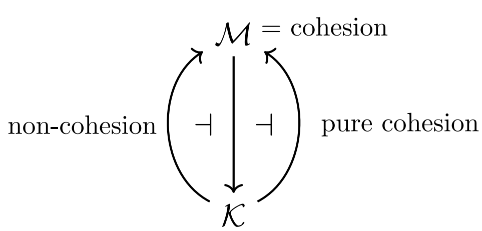
As I said in the beginning: the true foundations must serve mathematics, must provide a guide for finding appropriate conjectures and a guide for finding proofs of these conjectures. If it does not serve this end, it is not a foundation. Cantor made an advance towards such foundations and also Boole made an advance. But then someone applied an operation of the following sort: One takes these advances and declares them to be one's particular isolated speciality or expertise, then one arranges that one does *not* teach them, or one teaches that these are the only advances that exist or are possible, that they constitute **The Foundations** and you must believe that without them you cannot do your analysis and your geometry. To reinforce this belief one choses a notation and a mode for formalizing the concepts which is as different as possible from that used in everyday practice of mathematics, so that without a specialized study (into which you really have no wish to enter) you cannot understand it. Thus, briefly, the foundationalists direct the world of mathematics, but you poor devil can never understand their foundations. Naturally, such a foundationalist never enters into the study of newer developments in mathematics itself, because he knows already all that is possible and such "new" developments could only be some clever definition within their system or some very clever demonstration of a difficult theorem which is, after all, in the final analysis, merely a previously unnoticed consequence of their already known axioms.

That which I have just described we may succinctly refer to as Pure Foundations. But it should be immediately clear that from the point of view of mathematics

Pure Foundations
$$\cong$$
***Non*-Foundation**

As an example of the slightness of the connection between the purified foundation and the science of mathematics it suffices to consider the so-called definition of the "reals" as the power set of the natural numbers which they attempt to impose on the geometers and analysts for whom the real is a line and a system of continuos quantities.

Here we are attempting to use the expression "pure" and "non" in a consistent way to signify the philosophical relation known as unity and identity of opposites. For example, that which we have already discussed in terms of discrete and chaotic, might also be expressed as



since these two, while opposed as sub-categories as they are united in $\mathcal{M}$, are in themselves both identical with the category $\mathcal{K}$.

To illustrate in an extremely elementary example how the oppositeness of united identicals can be precisely expressed by adjointness, let us consider the problem of addition, as it is partly taught in elementary school. Every bookkeeper knows how to do a sum such as 17 + 8, one does (perhaps mentally) the following steps:

$$\begin{aligned} 17 + 8 &= (10 + 7) + 8 \\ &= 10 + (7 + 8) \\ &= 10 + [7 + (3 + 5)] \\ &= 10 + [(7 + 3) + 5] \\ &= (10 + 10) + 5 \\ &= 20 + 5 \\ &= 25. \end{aligned}$$

Here the seven steps can be explicitly justified either by the associative law or by the convention of positional notation, except for the crucial third step.

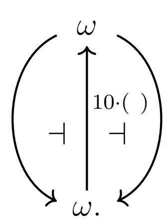
Foundations should also provide an explicitly teachable account of the justification of this third step which is sometimes described in an imprecise way as "striving for tens". Failure to arrive on your own at some formulation of the principle may contribute to the subjective conclusion that "I'm not gifted for mathematics" and hence objectively to the continuation of the frozen division of labor.

> Could this possibly have something to do with adjoint functors?
> Yes, and how!

As we know, the natural numbers under addition form a monoid $\mathbb{N}$, that is, plus is an associative operation with neutral element zero. But as with most synthetic operations the inverse or analyzing operation is not uniquely defined without further specification: this is precisely the problem (the ambiguity of decomposition of a given number 8) which must be resolved in order to render explicit the "necessity" of the above calculation. While $\mathbb{N}$ is of course itself a category, we must construct another category ω (actually by a so-called comma-category construction):

$$\text{Definition} \qquad x \leq y \quad \text{iff} \quad \exists z(x + z = y).$$

In the theory of potential this kind of definition of order is sometimes called the "specific" order. Every ordered set is a category in which the objects are the elements of the set and in which the morphisms are the particular order relation; this is a constantly used conception in the applications of category theory. Between such categories the functors are the maps which preserve the order. A crucial functor in our example is multiplication by 10, an opereation whose origin of course lies in the counting of people (who have 10 fingers each) and the passing to total number of fingers. This operation is not only order-preserving and full and faithful, but also has both left and right adjoints



Because of the full-and-faithfulness the two composites at the bottom are both the identity. This is a situation which we may call “an adjoint graph” which is partly dual to the “adjoint cylinder” in our previous example of unity and identity of opposites (involving Mengen, Kardinalen, pure cohesiveness and non-cohesiveness with the “points” functor as unifier “dual” to the unifying role of multiplication by 10 in the present example). The choice of the correct decomposition of 8 in our calculation is dictated by this adjoint graph. The next multiple of ten after seventeen is 20 which has a difference of three and hence the number whose difference with 8 is three, that is five, must appear. Thus one must somehow be aware of the central role of multiples of 10, as made precise by our adjoint graph and not only of the associative law, in order to begin the calculation. Not only that, but the precise relation between the synthetic operation and the analytic operation is also expressed by adjointness (not by mere inverse as in groups) for ω is in fact a closed category with $+$ as “tensor” and with truncated subtraction as “hom”:

$$\frac{x + y \geq z}{x \geq y \mathbin{\dot{-}} z}$$

Of course, what this analyzing “hom” measures in this example is mainly the degree of *lack* of the order relation; more precisely, this particular closed category satisfies a further important condition:

If $y \leq z$ then $y+(z-y) = z$, that is, the “evaluation map” $y+(z-y) \geq z$ is in fact an isomorphism in ω under the hypothesis $y \leq z$.

In this way we have precise explanations (which can be taught *when necessary*) for every step of such a calculation and is not only an a posteriori "justification" in terms of associativity alone.

Mathematics means "*that* science which it is possible to reliably teach". One knows that with the present system of teaching not all students are able to become bookkeepers, but armed with the conceptual instruments to explain more fully, we can begin to unfreeze that division.[3]

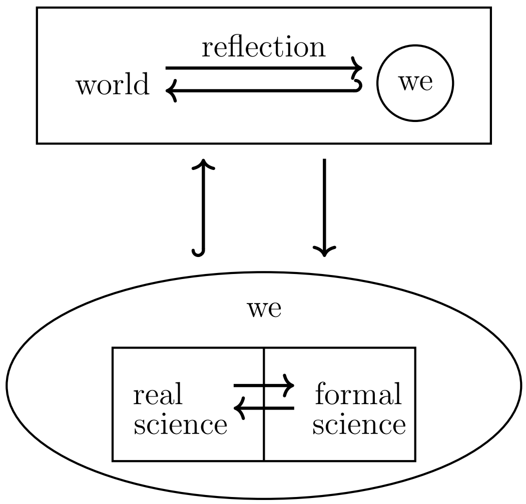
[3]And of course we have further similar examples of instruments; for with judicious use of the precise unity-and-identity-of opposites in categories and in 2-categories we can also render more explicit the basis of the laws of the dreaded differential calculus, which in 300 years the bourgeoisie have still not succeeded to teach to all workers, farmers, or bookkeepers. Perhaps they even avoided really trying to teach it, out of fear that the latter might understand and create an engineering on their own as well - at least my suspicion that the deProny principle is still in force was strengthened recently when the personnel director of a Swiss pharmaceutical firm, overhearing a plan to teach calculus to a wider audience, in all fervent seriousness responded: "criminals! sectarians! - such an idea if actually carried out would lead to the ruin of the finely tuned Swiss system of vocational training, and hence to catastrophe." For whom? ...

II

Wednesday, 14 June 1989

In 1844 a courageous German scientist wrote a book which is full of interest even for us today. In this work he applied the dialectical method to the study of a particular science and indeed in an extremely consistent way. On the other hand, he explicitly criticizes Hegel for his idealism. It is very easy to find this book in the library, but for "social" reasons it seems that almost no-one looked at it until recently. An essential part of the book is a description of the origin and foundation of the particular science in the reflection into our thinking of the real world. We see here an example of the unity of science which existed in that period for up to this point the foregoing description applies with equal justice to two great German scientists in that year and to two sciences, mathematics and political economy. The one which I want to concentrate on here is of course **Grassmann** and his excellent formulation of the foundations of mathematics. I am happy to explicitly thank Gian Carlo Meloni and Elisabetta Foresti for having urged upon me a couple of years ago the importance of a serious study of Grassmann's formulation of the foundation of our science.

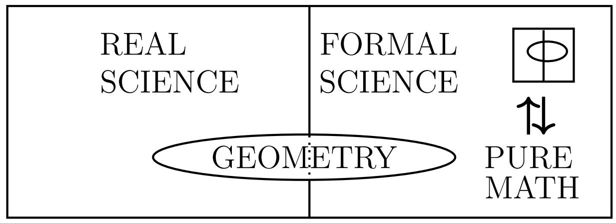
We are in the world, we can move ourselves in the world and hence the world is reflected in us. In our thinking we struggle to create an image of the world and in particular an image of ourselves, in our thinking process.

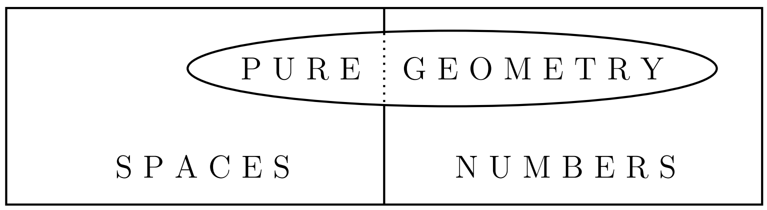


These two components of the scientific struggle are called by Grassmann "Real" Science and "Formal" Science, the sciences of things and of thoughts. In the latter, striving to find the laws of the development of thought there are two aspects:

- the philosophical science of dialectics which proceeds from the unity of the "general" to the "particular", and
- pure mathematics which proceeds from the particular thought to the general.

In particular, **Geometry** is not contained in pure mathematics, but is rather the application of pure mathematics (that is, of thoughts of a particular nature) to the real science of things in the world





In my second diagram I have permitted myself to continue the process of reflection by suggesting that the whole first diagram above is reflected into pure mathematics giving rise to particular ideas of spaces and numbers.

The fact that we are in the world and can move ourselves in the world is called "Space". The resulting reflections in thinking are called "Number" and the result of this fundamental relationship including measuring can be studied in more detail by means of both pure mathematics and dialectic. The pure geometry thus defined is the principal content of Grassmann's book and more particularly, he takes as fundamental the flat spaces and the extensive quantities and numbers. Sometimes flat spaces are called "affine", but there are many types of affinity and the terminology "flat" favored, for example, by Walter Noll, suggests more precisely what is intended. But more precisely,

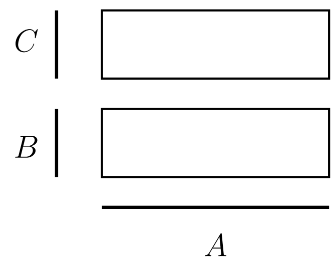
> how can we describe the general nature of these particular ideas
> of spaces and of systems of numbers?

I claim that we can make a considerable advance toward answering this question with the help of the theory of categories and especially by making explicit the distinction between two kinds of categories with finite limits, namely the distributive categories and the linear categories.

Moreover, it now seems that possibly the flat categories being studied by Carboni and by Faro and Schanuel may provide a precise description of the link between distributive categories and linear categories which must involve the notion of quantity; however, here I will concentrate on describing the distributive and the linear.

Both various categories of spaces in general, as well as various categories arising in the study of a particular space, have the special property of distributivity. In fact, for some purposes such categories may manifest themselves more precisely as toposes; toposes are in particular locally cartesian closed and local cartesian closure implies distributivity. But for reasons of presentation, both in the mathematical, as well as pedagogical senses, we emphasize here the more elementary, the more economical and hence still more general property of distributivity.

So let us first consider a category with finite coproducts denoted by + and, in particular, with the vacuous coproduct or initial objects denoted by 0, and with finite products denoted by $\times$ and 1. In any such category there is a canonical morphism $(A \times B) + (A \times C) \xrightarrow{\text{can}} A \times (B + C)$ which is not incorrect to picture according to our geometrical image as follows (provided that this canonical morphism is actually an isomorphism):



We also want that the canonical map $0 \to A \times 0$ is an isomorphism for all A. If our category $\mathcal{E}$ has not only finite products, but more general finite limits (equalizers, pullbacks, kernel pairs, inverse images) then for every object X the category $\mathbb{D}/X$ (the so-called comma category whose morphisms are all commutative triangles of $\mathbb{D}$ which end in X) is again a category with finite products (given by fibered products in $\mathbb{D}$ over X and having as terminal object the identity map 1_X of X from $\mathbb{D}$) and also with finite coproducts which are essentially "the same" as those of $\mathbb{D}$. Thus for our concept of distributivity we require that all the categories $\mathbb{D}/X$ for the various objects X of $\mathbb{D}$ should satisfy the invertibility as explained above of the canonical distributivity morphisms. But we require even more, namely the property sometimes referred to as disjointness of coproducts and strictness of the initial objects:

$$\mathbb{D}/0 \xrightarrow{\sim} \mathbf{1}$$

(the unique functor to the one-morphisms category) should be an *equivalence* of categories (with inverse given by 1_0), and moreover for any two objects X,

Y the functor

$$\mathbb{D}/X \times \mathbb{D}/Y \xrightarrow{\sim} \mathbb{D}/(X+Y)$$

(given by the functoriality of the coproduct in $\mathbb{D}$) should also be an equivalence, with inverse given by taking pullbacks along the coproduct injections. This last condition *excludes* distributive lattices from being distributive categories in our sense; however, any distributive lattice L gives rise to a distributive category $\mathbb{S}[L]$ by considering certain formal linear combinations having finite sets as coefficients – in particular, taking $L = 2$ reveals the "initial" distributive category to be $\mathbb{S}$ itself.

An entirely different sort of condition on a category is that it be linear. Again we consider a category with finite products and finite coproducts but we put immediately the condition

$$0 \xrightarrow{\sim} 1$$

that the initial and terminal objects be isomorphic, a condition which would immediately destroy any distributive category (since $\mathbb{D}/1 \xrightarrow{\sim} \mathbb{D}$ holds for any category). We can use this assumption to construct some further canonical maps. First note that any map from a coproduct to a product is uniquely determined by a rectangular matrix of "smaller" maps obtained by composing it with all relevant injections and projections.

$$\begin{array}{ccccc} & \sum_{i\in I} A_i & \xrightarrow{f} & \prod_{j\in J} B_j & \\ {}^{i}\nearrow & & & & \searrow^{j^*} \\ A_i & & \xrightarrow[f_{ij}]{} & & B_j \end{array}$$

(j^* is the j-th projection). Thus, in particular, when $0 = 1$, there is for any two objects A and B a canonical map

$$A + B \xrightarrow{\begin{pmatrix} 1_A & 0_{AB} \\ 0_{BA} & 1_B \end{pmatrix}} A \times B$$

whose components are either identity maps or the zero maps constructed as the composites

$$\underbrace{A \longrightarrow 1 \xleftarrow{\sim} 0 \longrightarrow B}_{0_{AB}}.$$

We say that a category is linear if all these canonical maps are isomorphisms so that in this precise sense finite coproducts and finite products are the same thing, (and for that reason sometimes called bi-products and denoted by a third symbol $A \oplus B$).

One of the most fundamental consequences of linearity is that for each object there is a canonical map

$$A^2 \xrightarrow{+} A$$

since in any case coproducts have canonical codiagonals

$$\begin{array}{ccc} A + A & \xrightarrow{\text{codiagonal}} & A. \\ \simeq \downarrow & \nearrow_{+} & \\ A \times A & & \end{array}$$

This operation is automatically associative and commutative and has 0_A as neutral element. Moreover, every morphism in the category automatically preserves 0, + and is hence "linear" in the sense commonly used by physicists and engineers. In particular, for any fixed object L the set of morphisms from $L \to A$ becomes a commutative monoid in the usual abstract sense with 0_{LA} as neutral element and with addition defined by composition as follows:

$$L \underset{g}{\overset{f}{\rightrightarrows}} A \qquad\qquad L \xrightarrow{\langle f,g \rangle} A \times A \xrightarrow{+} A, \quad L \xrightarrow{f+g} A$$

and composing with any given map $L' \xrightarrow{\lambda} L$ or $A \xrightarrow{\alpha} A'$ induces functorially a homomorphism of these abstract monoids.

Any linear category may thus be considered as a linear algebra with its morphisms a species of number. Note that a linear category can never be also distributive, unless it is zero: in the familiar example of vector spaces we see immediately that the dimensions of the left and right hand sides of the distributive law do not agree.

The fact that the coproduct in any category satisfies itself, the so-called law of the exchange of the middle two of four, follows from the universal property. Since the addition morphisms in a linear category are derived directly, in a unique way, from the coproducts the numbers in such a category satisfy the same law. This law, which in the presence of zero is equivalent to the conjunction commutativity and distributivity, itself expresses directly the fact that + is a self-homomorphism.

Given that the distributive categories are in general categories of spaces or of sheaves on a particular space and that the linear categories are categories whose morphisms form systems of numbers, both general and particular, we may now pose the general question of making explicit the relations between the two which arise in various branches of mathematics. It is still an open question of research whether these connections can all be seen as mediated quantitatively by the flat categories.

An important principle is

"a system of numbers **is** a Space".

This principle expresses itself mathematically as follows:

A given linear category $\mathbb{L}$ is often provided with an enrichment in a given distributive category $\mathbb{D}$. This means that there is a given functor

$$\mathbb{L}^{\mathrm{op}} \times \mathbb{L} \longrightarrow \mathbb{D}$$

also denoted by $\mathbb{L}$ and there is a given composition law

$$\begin{aligned} \mathbb{L}(A,B) \times \mathbb{L}(B,C) &\longrightarrow \mathbb{L}(A,C) \\ \mathbf{1} &\longrightarrow \mathbb{L}(A,A) \end{aligned}$$

which are natural morphisms *of the category* $\mathbb{D}$ such that moreover there is a natural bijection

$$\frac{\mathbf{1} \longrightarrow \mathbb{L}(A,B)}{A \longrightarrow B}$$

between the points of the $\mathbb{D}$-space $\mathbb{L}(A, B)$ on the one hand, and the actual $\mathbb{L}$-morphisms from A to B on the other hand, in such a way that the enriched composition law specializes on points to the composition law which makes $\mathbb{L}$ itself a category. Usually the product in $\mathbb{L}$ will be moreover itself $\mathbb{D}$-enriched, i.e. $\mathbb{L}(A, B_1 \times B_2) \xrightarrow{\sim} \mathbb{L}(A, B_1) \times \mathbb{L}(A, B_2)$ will be an isomorphism in $\mathbb{D}$.

Such an enrichment is precisely the fundamental structure of functional analysis, because it gives a determined meaning to continuous variations, smooth parameterizations, and approximations within systems of numbers all of which can be explained and studied in terms of $\mathbb{D}$-morphisms

$$T \longrightarrow \mathbb{L}(A, B)$$

where T is an appropriately chosen $\mathbb{D}$-space other than the point 1.

The objects of a linear category should be considered as *types* of numbers, with the morphisms $A \to B$ being the numbers of type $\frac{A}{B}$: in geometry these types may be simply linear dimensions, or various precise types of tensor bundles, whereas in physics they may be types such as mass, length, time, rate, pressure, energy, etc. In contrast, to the "general" role of functional analysis we must also consider many "particular" linear categories corresponding roughly to the fact that there are many particular spaces: algebraically expressed, we often associate to every *ring* R the linear category $\mathbb{P}(R)$ of all finitely-generated projective R-modules.

More precisely, the all important process of associating systems of numbers to spaces is often internally representable in the sense that a distributive category $\mathbb{C}$ of interest comes equipped with an enrichment of a small linear category $\mathbb{R}$; then with every object X of $\mathbb{C}$ we can associate another linear category whose objects are the same as those of $\mathbb{R}$, but whose morphisms are the $\mathbb{C}$-maps $X \to \mathbb{R}(A, B)$. As this category will typically have more idempotents that $\mathbb{R}$ itself, it is usual to take $\mathbb{C}_{\mathbb{R}}(X)$ the somewhat larger category obtained by splitting these idempotents. We thus obtain an essentially "representable" functor

$$\mathbb{C}^{\text{op}} \xrightarrow{\mathbb{C}_{\mathbb{R}}} \textbf{Lin cat}$$

with

$$\mathbb{C}_{\mathbb{R}}(X)(A, B) = \mathbb{C}(X, \mathbb{R}(A, B))$$

which we may with justice consider as a particular sort of "intensively" variable numbers. In particular, for $X = 1$ variability reduces to constancy

$$\mathbb{C}_{\mathbb{R}}(1) = \mathbb{R}.$$

It is clear by the universal property of $\mathbb{C}$-coproducts that

$$\mathbb{C}_{\mathbb{R}}(0) \cong \{0\}$$
$$\mathbb{C}_{\mathbb{R}}(X + Y) \xrightarrow{\sim} \mathbb{C}_{\mathbb{R}}(X) \times \mathbb{C}_{\mathbb{R}}(Y)$$

and by similar reasoning that for any $\mathbb{C}$-*epi*morphism $X' \twoheadrightarrow X$ we obtain a *faithful* linear functor

$$\mathbb{C}_{\mathbb{R}}(X) \hookrightarrow \mathbb{C}_{\mathbb{R}}(X')$$

(indeed for a "regular" epimorphism the latter is the inclusion of *those* variable quantities which are invariant for equivalence relation $X'' \rightrightarrows X$.)

In case $\mathbb{C}$ is cartesian closed, the representable notions of intensive variable number insert themselves immediately into functional analysis (as above defined). For the notion of T-parameterization in $\mathbb{C}$ used since the beginning of variational calculus

$$\mathbb{C}(T, \mathbb{C}_{\mathbb{R}}(X))(A,B) \underset{\text{def}}{=} \mathbb{C}(T \times X, \mathbb{R}(A,B))$$

is then representable in $\mathbb{C}$ as

$$\mathbb{C}(T, \mathbb{R}(A,B)^X).$$

The analogy between the disjointness property of distributive categories and the functorial property of representable intensive number may remind some of the Pythagorean principle "each thing is number" which, as suggested by Mayberry later in Cambridge, takes on not the usual idealist interpretation, but quite a materialist one, if we understand under "is" the reflection from reality to thought. This principle has, in fact, been used with varying degrees of explicitness by Galileo, Steiner, Cantor, Burnside and Grothendieck, for we can assign to each distributive category a rig of objects considered with the equivalence relation of isomorphism $\mathcal{B}(\mathbb{D})$ (where the $\mathcal{B}$ reminds us of Burnside and where "Rig" is a term suggested by Schanuel and me to signify that species of algebraic structure which if only it had negatives "n" would be that of commutative rings). But then for each object X of $\mathbb{D}$ we can consider $B(X) = \mathcal{B}(\mathbb{D}/X)$ and then $\mathbb{P}_{B(X)}$ will define a contravariant functor from $\mathbb{D}$ to **Lin cat** which takes coproducts to products. In particular, each point $1 \xrightarrow{x} X$ will induce an evaluation homomorphism $B(X) \to B(1) = \mathcal{B}(\mathbb{D})$ from variable numbers to constant ones. There is thus a partial representation $B(X) \to \mathcal{K}(\mathbb{D}(1, X), \mathcal{B}(\mathbb{D}))$ of a Burnside Rig as a Rig of functions on the cardinal of points of the space. If $\mathbb{D}$ is sufficiently rich, various portions of $\mathbb{P}_{B(X)}$ may even be representable in $\mathbb{D}$.

Even more directly spatial than the intensively variable numbers are the extensive ones considered by Grassmann, but whose study has been systematically kept in the shadows both before and since. These include both distributions and homology, but attempts to consider them as generalized "functions" (that is as intensive) can only lead to confusion and mystery. The extensively variable numbers can also be exemplified by "objective" numbers in the Burnside manner, for if we are given a category $\mathbb{A}$ with a coproduct preserving functor $\mathbb{A} \to \mathbb{D}$ so that $\mathbb{A}/0 = \mathbf{1}$ and $\mathbb{A}/(X+Y) \cong \mathbb{A}/X \times \mathbb{A}/Y$ for X, Y in $\mathbb{D}$ (for example a full sub-category such that $A_1 + A_2 \in \mathbb{A}$ iff $A_1, A_2 \in \mathbb{A}$) then $\mathbb{A}(X)$ defined to be the abstraction of the category $\mathbb{A}/X$ is

- an additive monoid which is
- a contravariant functor of X, i.e. $X \xrightarrow{\varphi} Y$ in $\mathbb{D}$ induces (by composition a coproduct-preserving functor $\mathbb{A}/X \to \mathbb{A}/Y$ and hence) a linear map $\mathbb{A}(X) \xrightarrow{\varphi_*} \mathbb{A}(Y)$
- moreover, this covariant functor satisfies

$$\mathbb{A}(X+Y) \xrightarrow{\sim} \mathbb{A}(X) \times \mathbb{A}(Y), \qquad \mathbb{A}(0) \xrightarrow{\sim} \{0\}.$$

In a general way the above three conditions constitute the minimum requirements for any particular notion of extensively variable number on $\mathbb{D}$. Instead of evaluation maps, extensive numbers include a *Dirac distribution*

$$\mathbb{A}(1) \xrightarrow{\delta_X} \mathbb{A}(X)$$

for each point $1 \xrightarrow{x} X$ of a space, and in place of the inclusion of constants in intensively variable numbers $B(1) \to B(X)$, the unique $X \to 1$ induces a *total*

$$\mathbb{A}(X) \xrightarrow{\delta_X} \mathbb{A}(1)$$

for example, the smoke B in the room X may have a density F with respect to the volume A in the same room, but has, in particular, a total. But remaining with the "objective" case we can also define a refinement of the intensive Burnside numbers in terms of their action as "densities" on the extensive ones, that is, if $\mathbb{B} \to \mathbb{D}$ is like $\mathbb{A}$ above, a "distribution" in $\mathbb{D}$, then an object F of $\mathbb{D}/X$ may be considered to be a proper ratio of $\mathbb{B}$-extensive numbers to $\mathbb{A}$-extensive numbers in X if there is a coproduct-preserving functor $\mathbb{A}/X \to \mathbb{B}/X$ agreeing with the pullback operation F^* on $\mathbb{D}/X$. By abstracting from these we can thus contravariantly associate to X a linear category whose objects are parameterized by the "distributions" $\mathbb{A}, \mathbb{B}$ etc. in $\mathbb{D}$.

Question by Meloni:

Can you clarify how the concept of functional analysis as you defined it, might apply in particular to combinatorial spaces?

Let us take for example $\mathbb{D} = \mathcal{K}_{\mathrm{fin}}{}^{\mathbf{\Delta}_1^{\mathrm{op}}}$ the category of finite reflexive graphs, that is, an object X is a diagram

$$\begin{array}{c} X \\ \partial_0 \left(\Big\uparrow \right) \partial_0 \\ X_0 \end{array}$$

of finite cardinals with both composites reducing to the identity on the points X_0, and let us consider a given $R \in \mathrm{Rig}(\mathbb{D})$ (there are many interesting examples of such, but for example we could consider a discrete or codiscrete one determined by an ordinary abstract Rig). Then as our linear category we could take $\mathbb{L} \underset{\mathrm{def}}{=} \mathrm{Lin}_R(\mathbb{D})$ the category of all R-modules A in $\mathbb{D}$. This means that both arrows and points of A can be separately added in a way that preserves the endpoint operations. For the enrichment we can define the equalizer

$$\mathbb{L}(A, B) \hookrightarrow B^A \rightrightarrows B^{A^2} \quad \text{in } \mathbb{D}.$$

Note that if R is actually a Ring, then we can define $\partial = \partial_1 - \partial_0$ and verify that $\partial^2 = 0$ and thus define $H(A) = Z(A)/B(A)$ in the usual manner. For example, taking $A = R^X$ in $\mathbb{D}$ we can define $H^*(X) = H(R^X)$ an intensive measure of the qualitative complication of the graph X which is of importance in the theory of electrical circuits and of the flow of traffic. The corresponding extensive measurement arises for example as $H(A)$ where $A = \mathbb{L}(R^X, R)$. $\partial(a) = 0$ means that the arriving flux is equal to the departing flux, whereas the boundaries are essentially quantities which depend only on the points and not on the arrows.

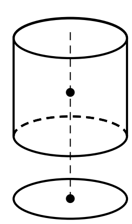
III

Friday, 16 June 1989

We now want to explain a partial solution to the problem of clarifying the contrast between the general distributive categories and the particular distributive categories. We will work mostly in the context of topos since there the technique is more fully developed and internal, although it should eventually become clear that most of the constructions and distinctions have sense for distributive categories at the expense of accepting sometimes a more external formulation.

We begin with the general theory of dimension. Normally one imposes at the outset the idea that dimensions must be minus infinity, zero, one, two, etc. but perhaps this idea is based on too limited an experience; we take instead the point of view that the possible dimensions are intrinsically determined by the category in which we work and form a structured ordered set of which it will be a theorem whether or not the order is total. The basic concept is again that of unity and identity of opposites UIO as expressed by a cylinder

$$\begin{array}{c} C \\ \Big\downarrow p \\ B \end{array}$$

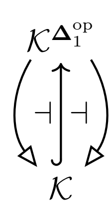


that is, by an arrow equipped with two preferred sections s_0, s_1 that is, $ps = 1_B$ for both s. In an arbitrary category the two sections of a cylinder are called "opposite" simply by formal definition. But in a 2-category such as

cat we can make a much more precise determination by requiring that these sections are adjoint to p, that is, $s_0 \dashv p \dashv s_1$ a situation which we may briefly denote by UIOA where the A stands for "Adjoint". In any cylinder C unites in itself the two opposites s_0 and s_1 which are however in themselves identical with the base B. Here one sees the importance of distinguishing (unlike in the usual set-theortical foundations) a subobject s from the object which is its domain.

Note. The apparently excessively simple mathematical formulation of the dialectical concept as a cylinder was first found in the context of adjointness in 2-categories, f. ex. in a forthcoming joint paper with Kelly, however, more recently I have applied it in the mere category of Rings to give a clarification of the foundations of differential calculus, as in a paper which I have previously circulated here. ("Unity and Identity of Opposites as exemplified by differential operator representors").

If $\mathbf{\Delta}_1$ is the three element monoid and if $\mathcal{K}$ is any regular category with co-equalizers, then $\mathcal{K}^{\mathbf{\Delta}_1^{\mathrm{op}}}$ participates in an adjoint graph in **cat** which has the further remarkable property that the left-adjoint preserves finite products (although not equalizers).



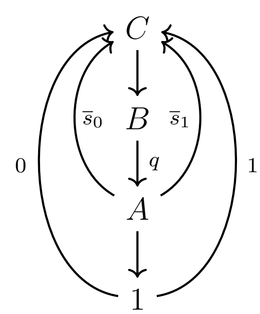
A simple, but important, example of a cylinder with a given base is the coproduct diagram

$$\begin{array}{c} B+B \\ s_0 \Big\uparrow \; \Big\downarrow p \; \Big\uparrow s_1 \\ B \end{array}$$

where the s are the respective injections and the p the co-diagonal. (This is actually initial in the category of all cylinders with base B). In this case, at least in a distributive category, the "unity" achieved by the cylinder is of a purely formal nature, but we might consider other examples where C is in fact connected, or at least that the fibers are connected unlike in this initial case where the fibers are a pair of disconnected points. Indeed, in homotopy theory there is a distinguished class of cylinders satisfying that sort of condition and a qualitative or homotopy invariant may be defined as any functor which carries the structural morphisms of a distinguished cylinder into isomorphisms.

An important condition emphasizing the oppositeness of a cylinder in a mere category is that the coincidence of the two preferred sections be empty, more generally, we can consider the equalizer E_1 of these two sections as a subobject of the base B. We could also consider E_2 the intersection of the two subobjects of C. There is obviously a natural map $E_1 \to E_2$ which without the unifying p would not necessarily be an isomorphism; however, an important exercise is to demonstrate that if the two monomorphisms s_0 and s_1 with common domain B indeed do have a common retraction p then $E_1 \xrightarrow{\sim} E_2$ is invertible, so that the two possible definitions of this coincidence in fact coincide. Note that our cylinders are not necessarily "trivial" where the latter refers to the following construction: If I is an object furnished with two preferred points t_0, t_1 then we have a cylinder with base 1; taking the cartesian product with any given B we obtain a constant or trivial cylinder $I \times B$ with base B.

Cylinders can be composed to give new cylinders but some compositions involve a qualitative leap. Let us now consider all the various cylinders which have a given total space C, and which are consistent with a given choice of a pair of preferred points denoted by 0 and 1.

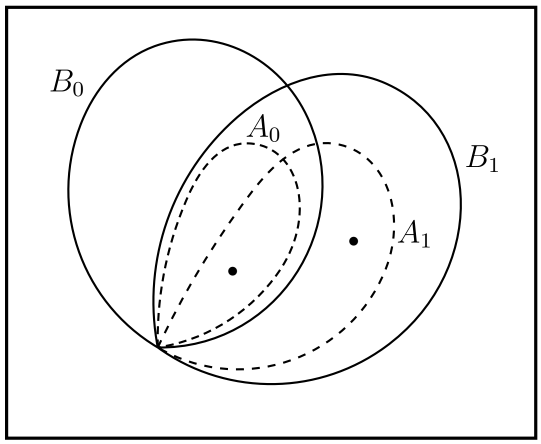


where most generally we say that p is consistent with $\overline{p}$ if there exists a (necessarily unique) q such that $qp = \overline{p}$ and also t_0, t_1 with $\overline{s}_k = s_k t_k$ for $k = 0, 1$ (note that the last pair of equations follow automatically in the case of adjoint cylindrifications of C and B).

But a condition qualitatively stronger than mere consistency is that p should already unite the opposites of $\overline{p}$ or as we may say that p is an "Aufhebung" of $\overline{p}$ if both of the ends $\overline{s}_0$ and $\overline{s}_1$ are contained in the right end of s_1. This additional equation

$$\overline{s}_0 = s_1 t_0$$

may be pictured in terms of subobjects of C as follows:



Note in particular that if we assume that our cylinders are strongly consistent in this sense with the bottom one then the point zero belongs to A_1 and not only to A_0 as mere consistency would requiere. If we consider C as a universal being then the things in C which have dimension A are those which are members of $\overline{s}_0$ whereas those which have dimension B are those which are members of s_0. We consider that the element 0 has dimension $-\infty$; those of dimension 0 are those which have dimension A for *all* the cylindrifications A which are "Aufhebungen" of $-\infty$. More generally, we could ask, given any A, for a smallest A' among all the "Aufhebungen" B of A, which we would then naturally call **the** Aufhebung of A. The latter does not exist, for example, in the case where C is the category of all presheaves on a finite category, as is exploited in my paper on display graphics. See the work of

Schanuel two years ago on the example of unbounded polyhedra for a case in which the lattice of these refined dimensions is not totally ordered. For a coarser, natural number value – dimension we may consider the length of chains of refined dimensions within which every successive pair stands in the Aufhebung relation. For those C in which the operation of "**the**" Aufhebung is well-defined, there is a single ascending chain which determines a coarse dimension number for each B. It can happen that the two "opposites" are actually entirely equal, for example in a pointed category, or with a Frobenius algebra.

Exponentiating a cylinder into a fixed value category gives rise of course to a graph but exponentiating a single map can often give rise by a Kan extension to an adjoint graph. For example, given an abelian category $\mathbb{L}$ if we consider the category $\mathbb{L}[d]$ of objects equipped with an endomorphism of square 0, then the induced inclusion taking $d = 0$ has both left and right adjoints which compose with the inclusion to give the identity on $\mathbb{L}$. In any such adjoint graph there is a canonical map from the left adjoint to the right adjoint, for suitable $\mathbb{L}$ we can take the image of this natural map obtaining a new functor H: in the abelian example just described this H is indeed the homology functor, as was explained by Cartan-Eilenberg 1956.

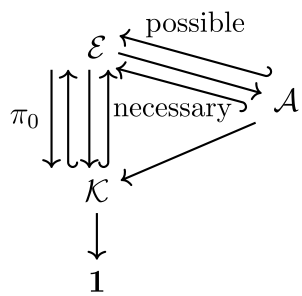
The topos of reflexive graphs has dimension 1 because, since the empty graph is also chaotic, the diagram

$$\begin{array}{ccc} & \mathcal{K}^{\mathbf{\Delta}_1^{\mathrm{op}}} & 1 \\ \text{discrete} & \uparrow \;\downarrow\; \uparrow & \text{chaotic} \\ & \mathcal{K} & 0 \\ & \downarrow & \\ & \mathbf{1} & -\infty \end{array}$$

is really aufgehoben and on the other hand one can show that there are no higher adjoint cylinders within it.

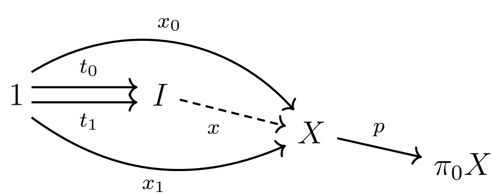
To see the next dimension consider instead the monoid $\mathbf{\Delta}_2^{\mathrm{op}}$ consisting of the ten order-preserving endomaps of the three-element totally ordered set and its topos $\mathcal{K}^{\mathbf{\Delta}_2^{\mathrm{op}}}$ of right actions. This is to be considered as a topos of combinatorial surfaces in which for example the Heyting algebra of truth-values is a surface with 17 triangular elements. Not only can this one be put on top of the graphs, in a cylindrical manner, with the left or skeletal adjoint being a kind of one-dimensional necessity operator and the right one or co-skeletal adjoint a kind of possibility operator, but in fact the discrete surfaces are also included in the possibility side, expressing the required Aufhebung relation. In both of these examples there is a further left adjoint to the discrete inclusion which is a product-preserving components functor, that is, giving the cardinality of the components an arbitrary triangulated surface.

If we know only the underlying graph A of a certain surface X then we know that X must contain the necessary $\mathrm{nec}(A)$ and must be contained in the possible poss(A). Much more generally then, given any category $\mathcal{E}$ we consider investigating its objects by means of higher and higher cylindrifications $\mathcal{A}$ of it, or dimensions.



In fact, Kelly showed that if $\mathcal{E}$ is a Grothendieck topos based on $\mathcal{K}$ then the class of all such $\mathcal{A}$ is a small co-Heyting lattice. Usually only dimension 0 admits the extra left adjoint π_0.

The functor π_0 counts the components of a thing X in terms of maps from X. But one also naturally thinks of measuring connectedness via becoming, that is, with help of maps x,



where I itself is a connected object and where t_0, t_1 are two specified "times": if such an x exists, with $xt_k = x_k$ one would naturally say that $px_0 = px_1$, i.e. that the two points lie in the same component of X. Now, intuitively, one would not have to use connected objects I of arbitrarily high dimensions to establish such connections: indeed, the highest dimension necessary for this would be philosophically "dimension 1". Thus it is fortunate that using adjointness alone we can prove the following theorem:

Theorem. *For any object X and for any cylindrification $\mathcal{A}$ of $\mathcal{E}$ if we denote by $\mathrm{sk}_{\mathcal{A}} X \to X$ the adjunction map associated with the left adjoint, then the induced map*

$$\pi_0 \, \mathrm{sk}_{\mathcal{A}} X \xrightarrow{\sim} \pi_0 X \quad \Longleftrightarrow \quad \mathcal{A} \text{ is an Aufhebung of } \mathcal{K}$$

Thus if there is a smallest such $\mathcal{A}$ we may consider it to be $\mathcal{K}_1$ the category of one-dimensional things.

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