Toposes in Geometry and Analysis*

F. William Lawvere

November 18, 2007

From the practice of Bernoulli, Volterra, Hadamard, Hurewicz, and countless others, it is clear that one of the most basic features of mathematical analysis is that the morphisms between two spaces constitute themselves a space. That spatial structure is determined by the right adjointness of exponential functors; their representability permits a unique notion of "functional" as a morphism whose domain happens to be such a space of morphisms.

Besides that axiom, the other definining axiom of the notion of topos is the existence of a subobject classifier (or truth-value object). (Grothendieck calls this space the "Lawvere element"; he had studied important parts of it under the name "Hilbert scheme".) Combining the two constructions one obtains the power-set-functor, which permits internalizing analogues of essentially all constructions of mathematical analysis; very agreeable exactness properties are also available as a consequence.

1. Generalized spaces parametrize variable sets

It is sometimes said that toposes arose historically as generalized spaces; but that is only partly true, and it is the other part that I want to emphasize here. It is true that since the 1930's there had been a need for a single category containing both topological spaces

*Prepared lecture for Journée Houzel. A different, but related, Talk was delivered on November 23, 2007, at the Institut Poincaré, Paris, for the celebration of Christian Houzel's 70th Birthday. See "The Role of History in Toposes in Geometry and Analysis" also in Lawvere Archive Posthumous Publications.

and groups on an equal footing; interpreting a space as its topos of sheaves, and a group as its topos of permutation representations, there is a common framework within which the algebraic topology of Hopf and Steenrod can be conveniently treated: Grothendieck defined a usefully restricted class of toposes (which he called étendue) that contains the union of the two; an étendue is locally isomorphic to a topological space, and certainly "locally (_)" is a time-honored strategy for generalizing. It turned out that (modulo the existence of points) the étendues are just those toposes for which there is a site consisting entirely of monomorphisms. As is well-known, it was the failure of the implicit function theorem which spurred the creation of a very significant kind of generalized space, the étale topos of a scheme, by means of Grothendieck's brilliant move 50 years ago that bypassed the lack of sections over monomorphisms to the base. The petit étale examples belong to the class of toposes, later studied by Peter Johnstone, in which the separable objects (i.e. objects A for which the diagonal in $A \times A$ has a complement) are adequate. These toposes have sites in which every map is epimorphic. There is a weaker cancellation law

$$(*) \qquad (fa = fb) \text{ and } ag = bg) \text{ implies } a = b$$

satisfied by both categories of epics and categories of monics, so that toposes having a $(*)$ site include both Grothendieck's étendue and Johnstone's adequately separable toposes. Thus we surmize that toposes of "variable sets" have subcanonical sites with no non-trivial idempotents. That implies a still more general condition which does not mention sites:

$$(**) \qquad \text{the topos itself has no central idempotents.}$$

(Recall that the center of a category is the monoid of natural endomorphisms of the identity functor.)

In contrast with the above kinds of "generalized space" (i.e. a topos parameterizing "variable sets") the other historically arising topos were categories of "cohesive spaces": in the study of these, central idempotents (effectively "degenerations") do occur.

2. Logic as a needed auxiliary tool in geometry

It is also sometimes said that most of the post-Grothendieck work on toposes is mere logic. In reality, the pursuit of geometry and analysis necessitated a certain explicit attention to logic; as a by-product some new light was shed on traditional questions of logic, so that for example it was in collaboration with the algebraic topologist Myles Tierney that the topos version of the independence of the continuum hypothesis was proved, and it was in topology that Heyting lattices emerged independently of any "constructivist" philosophy. If, however, we interpret "Logic", not only in the narrow sense of the science of proving statements about fixed concepts, but in the traditional philosophical sense of the science of dialectically constructing concepts, then clearly much of category theory is Logic.

3. Cohesion of categories of space, having homotopy & functional analysis

Cantor initiated his set theory by negating the perceived cohesion of the "Mengen" occurring in geometry and analysis; using the method that he borrowed from the Swiss geometer Jakob Steiner, he compared spaces by isomorphisms in a category but, in contrast with Steiner, he insisted that there be "keine Stetigkeit" (no condition of continuity) in his morphisms. He saw rather soon that this insistence on absolute non-cohesiveness would imply the generalized continuum hypothesis for his power-set-functor. The later development of set theory neglected this initial vision of Cantor; the vision of von Neumann's hierachy is more complicated but not powerful enough to imply such a basic conclusion. By contrast, mathematical practice has internalized the cohesion-free sets and performed the double negation that uses them as a background recipient for all kinds of mathematical structure, including structures that model various needed categories of cohesion. Often a whole category of cohesion, including infinite-dimensionl spaces, etc. can be generated form a tiny "germ" of it, such as the unit interval or a generic convergent sequence; a long-neglected observation of Euler permits a qualitative deepening of that dialectic, as we will see. Using the traditional interval I as a germ leads to a topos with ungeometric properties such as Peano's space-filling curve and the rejection of a total ordering property of the distributive lattice I; a more "tame" notion of continuous map $I \to I$ is still needed in order to remedy those defects, but meanwhile the Fréchet-Fox-Johnstone topos generated by a convergent sequence should be used more.

The default model of cohesion is often considered to be the category of topological spaces, but the well-known defects of that as a background for algebraic topology and for functional analysis began to be recognized and treated in 1945 by Fox and by Mackey, using sequential convergence and bornology, respectively; in 1949 by Hurewicz, using his k-spaces; and in 1950 by Eilenberg-Zilber, using simplicial sets. As I urged ten years ago in the Rendiconti di Palermo, it is useful to consider models of cohesion which are themselves toposes; that seems to require an approach to geometrical cohesion based on figures and incidence relations, an approach which can be discerned in Volterra's use of the term "element".

Perhaps the earliest example of such a cohesive topos is the one whose spaces are just simplicial sets. Eilenberg and Zilber in their 1950 paper introduced them by observing that the "points" functor is not faithful. This is the classifying topos for "intervals" (= totally ordered distributive lattices) but surprisingly, in spite of its constant application, very little is explicitly exploited of its very powerful particular properties qua topos. Quite fertile as an example is the topos of 1-truncated simplicial sets, or reflexive graphs (which is clearly not a generalized space, because it has the degeneracies typical of the other kind of topos).

In this other part of topos theory, a key step was Grothendieck's 1960 treatment of the category of cohesion known as analytic spaces. Its algebraic geometry analogs in 1964 and its C-infinity analogs in 1967 were recognized as toposes. The intrinsic notion of functional in these toposes is essentially that of Volterra's analytic functionals, which made more explicit the concept used in the calculus of variations for 300 years.

4. Axiomatic cohesion

To analyze cohesion, we can approach Cantor's negation from a synthetic standpoint. Thus, suppose that we have a category $\mathcal{E}$ of cohesion and a category $\mathcal{S}$ of non-cohesion or discrete spaces. With Cantor we assume that $\mathcal{S}$ is given as a coreflective subcategory of $\mathcal{E}$, that is, that every space has a maximal discrete core. (It should not be expected that this points functor is faithful.) At the opposite end of $\mathcal{E}$ is the right adjoint to the points functor, the inclusion of the subcategory of anti-discrete or chaotic spaces

$$\begin{tikzcd}
\mathcal{E} \arrow[d, ""{name=M}] \\
\mathcal{S} \arrow[u, bend left=60, ""{name=L}] \arrow[u, bend right=60, ""'{name=R}]
\arrow[from=L, to=M, phantom, "\dashv"] \arrow[from=M, to=R, phantom, "\dashv"]
\end{tikzcd}$$

The two opposed subcategories often have scant intersection, though they are identical in themselves. Assuming that $\mathcal{E}$ is a topos, these three functors make $\mathcal{S}$ an essential subtopos. Grothendieck's toposes are characterized by the fact that $\mathcal{S}$ is chosen to satisfy Cantor's intuition of non-cohesion and non-variation to the extent that epimorphisms split and there are only two subobjects of the terminal object. However, Grothendieck's powerful relativization principles assures rich content from the consideration of more general bases $\mathcal{S}$; already in algebraic geometry a less extreme form of "discreteness" is suggested by, and fundamental to, Galois theory.

A subcategory $\mathcal{C}$ of $\mathcal{E}$ can be defined to consist of all spaces A for which the diagonal map

$$(***) \qquad S \to S^A$$

is an isomorphism for all discrete spaces S, for example, $A = 1$ and products $A_1 \times A_2$ are in $\mathcal{C}$ if A_1, A_2 are; but $A = 0$ and $A = 2$ are not in $\mathcal{C}$ if $\mathcal{S}$ is non-trivial. For any space X, and any A in $\mathcal{C}$, a morphism $A \to X$ is called a "figure in X of shape A" and the slice category $\mathcal{C}/X$ of all these A over X is called "the geometry of figures and incidence relations in X" (obviously we refute the rumors that in category theory one cannot get inside the objects). Any morphism $X \to Y$ in $\mathcal{E}$ is "continuous" in the sense that it maps figures to figures without tearing the incidence relations. Dually, a map $X \to A$ is called a "function of type A on X" and the co-slice category $X/\mathcal{C}$ of A's under X is called "the algebra of functions on X". Because the product of two types is a type, the algebraic operations on functions include all binary, ternary, etc. morphisms on the types in $\mathcal{C}$. Any morphism $X \to Y$ induces a homomorphism in the reverse direction of the functions algebras.

The Galois connection induced by the relation (∗∗∗) conversely defines, for a given class of objects A in a topos $\mathcal{E}$ a subcategory $\mathcal{S}$ of all the objects which look discrete to them. (Zero is discrete if it is not among the A's.) If S looks discrete to A, then so do S^X for all X. Indeed, the discrete inclusion preserves all projective limits that exist in $\mathcal{S}$, so if $\mathcal{S}$ is complete enough relative to $\mathcal{E}$, there is a left adjoint π_0 that preserves finite products and extends the adjoint string to four. Strikingly, it is often the case that the entire topos $\mathcal{E}$ can be reconstructed from just one or two objects A and the corresponding discrete objects. For example, Euler's remark that the continuum consists of ratios of infinitesimals takes the following concrete form in algebraic geometry, analytic geometry, and smooth geometry: There is a very tiny object D with just one point which is contractible in the sense that $A = D^D$ perceives as discrete the same spaces S that look discrete to D. Every space X is determined by the $\mathcal{S}$-valued presheaf on $\mathcal{C}$ whose value at any A is the underlying discrete space of X^A. In particular, X^D is the tangent bundle of X, and $X^{(D^D)}$ has the space of paths in X as a retract. In fact, all spaces X are determined by a drastic restriction of these presheaves to a subcategory $\mathcal{C}_0$ consisting of a few A's of the kind D^D. A stronger form of "infinitesimal generation" requires that there be no proper subtopos of $\mathcal{E}$ containg D; any such topos would automatically contain D^D, etc. In the case of algebraic geometry over a field that is not algebraically closed, the resulting canonical notion of "discrete" is not as extreme as Cantor's, but is rather nearer to Galois' notion of point, involving sheaves S on the category of finite field extensions. The structure of the function algebras automatically includes a ring structure because, for example, the submonoid R of the tangent bundles of D consisting of tangents at the base point has not only its intrinsic multiplication, but also a uniquely determined addition; indeed R is the usual line in the examples. The particularity of the monoid R^R largely serves to distinguish various $\mathcal{E}$ over the same $\mathcal{S}$, extending results of Hartog and Hadamard. The subspaces $\mathrm{Hom}(R^X, R)$ of the double exponentials play the role of distributions of compact support (which are clearly not "generalized functions" since they transform covariantly, rather than contravariantly, with the domain space). In fact, there is a whole intrinsic smooth functional analysis in such a topos, which has been applied by Kock and Reyes to the wave equation and the heat equation.

Differential equations themselves form a topos. For first-order equations that is almost obvious: indeed for any pointed object D, a D-vector field $X \to X^D \to X$ on an object X is equivalent to a pointed action, and using the free monoid on D, the category of all such actions in $\mathcal{E}$ is seen to be a topos, having an essential morphism

$$\mathcal{E} \to \mathcal{E}^D$$

with faithful inverse image = the underlying state space functor and with "chaotic" forward image. In this case the adjointness of these functors is enriched in $\mathcal{E}$ itself. But there are also many such faithful functors $\mathcal{E} \longleftarrow \mathcal{E}^\alpha$ with both adjoints where the adjointness is merely $\mathcal{S}$-enriched. The functor might then be called "underlying configuration space". Such "higher monoids" go back to Fibonacci, who made mathematical the idea of state that "the old is pregnant with the new" and to Galileo, who found such "states of becoming" in mechanics and associated them with inertia. For example, consider $\alpha : D \to D^2/2! = A$ as the two coalesced axes. Then by an action of α on X is meant a section of $\alpha^* : X^A \to X^D$, that is a given rule prolonging any first-order path to a second-order path. The category $\mathcal{E}^\alpha$ of such actions, or α-order laws, has solutions paths as its morphisms in the case that the domain models a time interval; the general morphisms carry solutions to solutions. The category $\mathcal{E}^\alpha$ will be a topos, with $\mathcal{S}$-adjoint essential morphism $\mathcal{E} \to \mathcal{E}^\alpha$, in case the object D is an ATOM ("amazingly tiny objectified motion") in the sense that there is a topos morphism $\mathcal{E} \to \mathcal{E}/D$ right adjoint to the usual $\mathcal{E}/D \to \mathcal{E}$. That is indeed the case for the typical D = spectrum of the dual numbers, in the various smooth toposes. If we consider the much simpler $\alpha : D \to 1$, then the "α-th order laws" forbid any motion, so that $\mathcal{E}^\alpha = \mathcal{S}$.

Assuming that the discrete subcategory is Galois-closed (though not necessarily co-infinitesimal), the left adjoint π_0 to the inclusion must preserve finite products. We can therefore define a new $\mathcal{S}$-valued category with the same objects as $\mathcal{E}$ but with

$$[X, Y] = \pi_0(Y^X)$$

as homs. This new Hurewicz category $H(\mathcal{E})$ is not a topos but does have sums, products and exponentials. Any contractible object in $\mathcal{E}$ becomes isomorphic to 1 in $H(\mathcal{E})$. The "points" $1 \to Y$ in the sense of $H(\mathcal{E})$ are just components of Y in $\mathcal{E}$; we therefore expect that the discrete inclusion of $\mathcal{S}$ in $H(\mathcal{E})$ will have isomorphic left and right adjoints. That is true in continuous and finite kinds of category $\mathcal{E}$, but typically not for combinatorial kinds of $\mathcal{E}$, which is part of the reason "weak" equivalences arise in combinatorial homotopy theory.

5. Quality, extensive and intensive

In those cases where the isomorphism just described holds, we say that homotopy type (or "form") is an "extensive quality" over $\mathcal{S}$, in the sense that it is a functor

(1) to a category $\mathcal{Q}$ over $\mathcal{S}$ having collapsed string of adjoints,

that

(2) is compatible with the left adjoint "components" part of the uncollapsed string under its domain $\mathcal{E}$ over $\mathcal{S}$.

On the other hand, an "intensive quality" over $\mathcal{S}$ is a functor, whose codomain likewise has a collapsed adjoint string to $\mathcal{S}$ but which is compatible instead with the right adjoint "points" functor in the string under its domain $\mathcal{E}$. The simplest codomain category $\mathcal{Q}$ for qualities is $\mathcal{S}^e$, the actions of a single central idempotent e (also known as split epimorphisms). In fact, any non-trivial quality $\mathcal{Q}$ over $\mathcal{S}$ has $0 \neq 1$ in its monoid of central idempotents, and conversely. The canonical intensive quality for general $\mathcal{E}$ clearly ought to involve the subcategory of all those spaces L for which the map $\mathrm{pts}(L) \to \pi_0(L)$ is an isomorphism.

I will assume the weak Nullstellensatz which requires that the analogous morphism $\mathrm{pts}(X) \to \pi_0(X)$ is at least an epimorphism for all spaces X. Then the L for which this epimorphism is invertible from an essential quotient topos of $\mathcal{E}$ that is obviously a quality; it is suggestive to call it "substance". In case $\mathcal{E}$ is the topos of reflexive graphs in $\mathcal{S}$, the resulting intensive quality is $\mathcal{S}^e$, interpreted to consist of graphs whose edges are all loops. The two adjoints can be thought of as super-rarefication and super-compression, revealing the substance of a general X in the following sense:

We can find the "quantum numbers" at each point (for example by counting loops in a graph X) even though rarefication leads to negligible interactions between points, whereas at the opposite condensed extreme all the interactions in X contribute to the quantum numbers of a few super-molecules (for example, the components of a graph, except that all degenerate loops in a given component are identified). The canonical compression map between the two adjoints gives additional partial information about the organization of the substance within the sample X (for example what part of the condensed substance is virtual in the sense that it comes from interactions in X, rather than from the pure substance as revealed in the rarefied condition).

Because intensive quality can include the property of being an n-dimensional manifold, the Poincaré conjecture illustrates that some non-trivial spaces are determined by their intensive and extensive quality. There are combinatorial examples of that phenomenon, due to Schanuel.

6. Bornological and combinatorial cohesion

A rather different sort of cohesion is typified by bornology. Professor Houzel was one of the first to discover that toposes and bornological spaces interact powerfully in geometrical and functional analysis, and it was from the study of his papers that I arrived at several of the topics under discussion. No matter whether one agrees with Dieudonné or Volterra on the true framework for functional analysis, one can always define boundedness of a family of linear operators to mean their equi-continuity, and this works well as an enrichment into the cartesian-closed category of bornological sets. This leads to two proposals:

(1) Take bornological sets as the framework, and in fact

(2) replace the category of bornological sets by its topos completion.

That topos appears in SGA4 as a counter example, showing that a left exact Yoneda embedding may not preserve coequalizers, even when it preserves sums and epimorphisms.

Specifically, the category of countable sets embeds into the topos $\mathcal{E}$ consisting of those of its presheaves X for which $X(0) = 1$ and $X(A + B) = X(A) \times X(B)$ for any two countable sets A and B. This embedding tautologically preserves products and sums, and preserves epimorphisms because by the axiom of choice they split; however, the orbit space of the successor map on natural numbers is trivial in the site, yet involves a whole Fréchet filter in the topos $\mathcal{E}$. There are two very distinct natural number objects that have the same points and the same (uncountable) monoid of endomaps; one is a sub-semiring of the other, yet the only maps from the larger to the smaller have finite image. As in any Grothendieck topos, there is a canonical real number object R constructed by internalizing the notion of Dedekind cut; in this case R turns out to have ordinary reals as points and the usual bounded sequences as figures. The category $\mathrm{mod}(R, \mathcal{E})$ is then a Grothendieck AB5 abelian symmetric monoidal closed category (as for any commutative ring in a topos) but contains as a full subcategory all the bornological vector spaces in which boundedness is determined by sequences, for example, Fréchet spaces.

The rather special sheaf condition involved in the construction of the bornological topos also occurs in combinatorial topology and set theory. Consider a category $\mathcal{F}$ based on $\mathcal{S}$ by a five-fold string of adjoint functors (actually just a united adjoint pair of essential points, 0 and 1); for example, $\mathcal{F}$ could be the category of presheaves on a site $\mathcal{S}$ that has distinct initial and terminal objects, such as the category of finite sets, or the category of countable sets. Somewhat opposite to the previous construction of intensive quality as a quotient topos where two downward functors coalesce, we consider now sub-toposes where the canonical map between upward adjoints become partially equal, in other words, the two opposites overlap to a larger extent than just $\{1\}$, because the leftmost has been reflected into the sub-topos. If the overlap includes a topos (that is, a subcategory of $\mathcal{S}$ closed under exponentiation that has its own smaller truth value object) then the sub-topos $\mathcal{E}$ of $\mathcal{F}$ that results is of interest in the study of the needed layers of set theory. If the overlap consists only of finite sets, this $\mathcal{E}$ is just the bornological topos (in the case where the site is the category of countable sets); the discrete bornology on a set has only finite sets bounded, whereas the co-discrete bornology has all subsets bounded, showing clearly where the two opposite inclusions agree. The smallest possible enlargement of the overlap, namely adjoining only 0, has far-reaching consequences, because the resulting sub-topos $\mathcal{E}$ (in case the site consists of just all finite sets) is the combinatorial topos completion of the classical category of simplicial complexes. Contrary to $\mathcal{F}$, wherein all connected objects are contractible, here all homotopy types are represented because the new π_0 does not preserve equalizers, even though it preserves products. Analogously to the category of simplicial sets, which contains the generic distributive lattice (and in fact all small categories as a reflective subcategory) this combinatorial topos $\mathcal{E}$ contains the generic Boolean algebra and also all groupoids as reflective subcategories (the combinatorial Poincaré groupoid of any simplicial complex is an instance of the reflection). Because a distributive lattice embeds canonically in a Boolean algebra in any topos, the appropriate geometrical realization for this classical combinatorial topos $\mathcal{E}$ is based, not on an interval, but on the infinite-dimensional sphere as the fundamental contractible object.

We give our sincere gratitude to Francisco Marmolejo for his time and effort to create a refined and polished version that looks amazing.

The Archive Family