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Sheaves, Large and Small

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Abstract. The "petit" toposes of sheaves on particular generalized spaces are described and related to the particular "gros" toposes of general spaces of the simplicial, bornological, and algebraic categories. The topos-theoretic image construction is described and used in a particular case as a foundation for general topology. Some indication of the historical process, from Hopf 1940 through Leray 1945 and Grothendieck 1958 to Grothendieck's 1960 study of central examples of both kinds of toposes is given, as are some suggestions for further use of toposes in functional analysis and dynamical systems.

1. Algebras of set-valued continuous functions on spaces and other situations

A sheaf is a continuous set-valued function on a "topos". A topos is a very general sort of "situation", for example a generalized space. As in analysis and geometry, where complex-valued rather than set-valued functions are considered, we start with the algebra of functions and then try to understand the actual situations by dualizing. For lack of a dual name these algebras are also called *toposes*. They are special *categories*: for example the algebra of *sheaves on the one-point situation* is the category $\mathcal{S}$ of all abstract sets and mappings. Even though the scalars are sets rather than complex numbers we can add and multiply, since these algebras are categories with coproducts and products. The following distributive law is valid

$$A \times B_1 + A \times B_2 \xrightarrow{\sim} A \times (B_1 + B_2)$$
$$0 \xrightarrow{\sim} A \times 0$$

where 0 is the empty coproduct or "initial object" of the (category of sheaves on a) given topos. Even better, we have $\varinjlim$ over any arbitrary indexing category $\mathbb{I}$, generalizing the $+$ (which corresponds to $\mathbb{I}$ = discrete 2), and we also have pullbacks (and equalizers, inverse image, binary intersection) "fibered products" generalizing $\times$ and the bilinearity or distributivity holds of all these

$$(\varinjlim_{\mathbb{I}} A_i) \times_C (\varinjlim_{\mathbb{J}} B_j) \xleftarrow{\sim} \varinjlim_{\mathbb{I}\times\mathbb{J}} (A_i \times_C B_j)$$

where all A_i (and B_j) are assumed compatibly equipped with maps $A_i \to C$, $B_j \to C$. Here the $\mathbb{I}$, $\mathbb{J}$ are any small index categories; if $\mathbb{I}$ happens to be *filtered* (directed) then $\mathbb{I} \xrightarrow{\text{diag}} \mathbb{I}\times\mathbb{I}$ is cofinal, so as a consequence we have that $\varinjlim_{\mathbb{I}}$ commutes with fibered products (hence binary products, equalizers, ...) for filtered $\mathbb{I}$. Together with the condition that the whole algebra (a class) has a *set* of generators, these were essentially the axioms introduced in May 1963 by Grothendieck and Giraud. All these distributivities are actually consequences of the existence of a few adjoint functors, so Tierney and I in August 1970 (ICM) were able to show that equivalent axioms are (as improved by Mikkelsen and Paré slightly later)

there exist internal "function spaces" satisfying a natural bijection

$$\frac{X \longrightarrow Y^A}{A \times X \longrightarrow Y}$$

there exists a "truth-value" object Ω which classifies arbitrary monomorphisms by unique "characteristic functions"

$$\frac{X \longrightarrow \Omega}{? \longrightarrow X}$$

But the definition of morphism between $\mathcal{S}$-toposes remains

$$\mathcal{X} \xrightarrow{f} \mathcal{Y}$$

"is" a functor f^* from the algebra of $\mathcal{Y}$-sheaves to the algebra of $\mathcal{X}$-sheaves which preserves "addition and multiplication" i.e. $\varinjlim_{\text{arb}}$ and $\varprojlim_{\text{fin}}$. By the "small generator" assumption, the preservation of $\varinjlim$ is equivalent to the fact that f^* has a right adjoint f_* from $\mathcal{X}$-sheaves to $\mathcal{Y}$-sheaves; hence we can identify f with a pair of fuctors f_*, f^* such that

$$f^* \dashv f_*$$
$$f^* \text{ is lex}$$

where lex means "left exact", i.e. preserves $\varprojlim_{\text{fin}}$, for which suffices the preservation of the empty product (or terminal object) 1 and of fibered products.

The older examples (Leray, Godement, ...) were values of the 2-functor

$$\mathbf{top} \xrightarrow{\text{sh}} \mathbf{Top}/\mathcal{S}$$

into the category (of categories) defined above, defined on the classical category **top** of topological spaces and continuous maps (explained below). Even easier is the 2-functor

$$\mathbf{gps} \longrightarrow \mathbf{Top}/\mathcal{S}$$

which assigns to every group the category of all G-sets (exercise: compute Ω_G and Y^X where X, Y are G-sets). *One* of the reasons for $\mathbf{Top}/\mathcal{S}$ is the need, arising from the Hopf 1940 discovery in algebraic topology (which led to the cohomology of groups and hence to homological algebra) for having both **top** and **gps** in one bigger category.

The construction of $\text{sh}(X)$ for a space X (and basis $\mathcal{U}$ of open sets) can itself be understood in terms of a map of "simpler" toposes

$$\mathcal{S}^{|X|} \underset{\Gamma_*}{\overset{\Gamma^*}{\rightleftarrows}} \mathcal{S}^{\mathcal{U}}$$

where the left is just the category of all families A of sets abstractly indexed by the points of X, while the right is the category of all "presheaves" B on the poset $\mathcal{U}$. Thus B involves a set $B(U)$ given for each $U \in \mathcal{U}$ but also a set map $B(U) \to B(U')$ given for each pair $U' \subseteq U$ in $\mathcal{U}$, subject to obvious functoriality (transitivity) conditions. A map $B \to B'$ is *natural*, i.e commutes with these given "restriction" operations. Then

$$\Gamma_*(A)(U) \underset{\text{def}}{=} \prod_{p \in U} A_p$$

$$\Gamma^*(B)_p \underset{\text{def}}{=} \varinjlim_{U \ni p} B(U)$$

and it is easily checked that $\Gamma^* \dashv \Gamma_*$, i.e.

$$\frac{B \longrightarrow \Gamma_* A}{\Gamma^* B \longrightarrow A}$$

The all-important "lex" condition on Γ^* follows from the fact that the $\varinjlim$ is over all the $\mathcal{U}$-neighborhoods of p, which form a *filtered* poset. Note that the composite $G_\Gamma = \Gamma^* \circ \Gamma_*$ is a glorified *interior* operator on $\mathcal{S}^{|X|}$, extending the well-known one on $\mathbf{2}^{|X|}$ by considering those A such that $A_p = \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right.$.

The category $\mathrm{sh}(X)$ is the $\mathbf{Top}/\mathcal{S}$ *image* of Γ, and as such it can be described from either end; in the factorization

$$\mathcal{S}^{|X|} \xrightarrow[p]{} \mathrm{sh}(X) \xrightarrow[i]{} \mathcal{S}^{\mathcal{U}^{\mathrm{op}}} \qquad\qquad ip = \Gamma$$

the distinctive properties are

p^* is *faithful,* such p are often called "surjective"

i_* is *full and faithful,* so can be essentially regarded as a full sub-category.

The condition that B "is a sheaf", i.e. lie in the subcategory, turns out to be the famous "pasting" condition satisfied by examples such as

$$B(U) = \mathbf{top}(U, Y)$$

where Y is any other fixed topological space. The *subsheaves* of 1 are just the open sets of X, but the sub-presheaves $R \subset U$ of such U in $\mathcal{S}^{\mathcal{U}^{\mathrm{op}}}$ are determined by a family $\mathcal{V} = \{V \in$

$\mathcal{U}|V \hookrightarrow R\}$ of elements of $\mathcal{U}$ for which

$$V \in \mathcal{V} \Longrightarrow V \subseteq U$$
$$V' \subset V \in \mathcal{V} \Longrightarrow V' \in \mathcal{V}.$$

Those R for which $\Gamma^*(R) \xrightarrow{\sim} \Gamma^*(U)$ are called *coverings* of U. Then the pasting condition is just

$$\begin{tikzcd} R \arrow[r, hook] \arrow[dr, "f"'] & U \arrow[d, "\bar{f}"] \\ & B \end{tikzcd} \qquad \begin{array}{l} \forall \text{ coverings } R \text{ of } U \\ \forall f \\ \exists! \bar{f}. \end{array}$$

Note that f itself is any *natural* map (i.e. in $\mathcal{S}^{\mathcal{U}^{\mathrm{op}}}$); it is determined by f_V for each $V \subseteq R$, $V \subset U$ subject to compatibility for any $V' \subseteq V$.

The order-ideal property of $\mathcal{V}$ eliminates the need to consider "overlaps" $V_1 \cap V_2$ separately.

The maps $\mathrm{sh}(X) \to \mathrm{sh}(Y)$ of toposes are all uniquely determined by continuous maps $X \to Y$, at least if X, Y are Hausdorff spaces; even for the more general "sober" spaces, we may consider that sh determines **top** $\to$ **Top** as a subcategory.

A topos $\mathcal{E}$ is said to *locally* lie in some category of **Top** if there is $S \in \mathcal{E}$ with $S \to 1$ epimorphic so that $\mathcal{E}/S$ (the category whose maps are commutative $\mathcal{E}$-triangles over S – it is also a topos) lies in the subcategory. In particular, $\mathcal{E}$ is an *étendue* if it is locally in **top**.

Any group G "is" an étendue, for if $\mathcal{E} = \mathcal{S}^{G^{\mathrm{op}}}$ and $S = G$ acting on itself by right multiplication, then it is easily calculated that $\mathcal{E}/S \approx \mathcal{S} = \mathrm{sh}(1)$.

This apparently trivial remark is actually important. For some more general examples, note first that for any small category $\mathbb{C}$, $\mathcal{S}^{\mathbb{C}^{\mathrm{op}}}$ is a topos since, from the Giraud-Grothendieck point of view, all the distributive laws are valid because $\varinjlim$ & $\varprojlim$ are computed $C \in \mathbb{C}$-wise in this case and the laws are valid in $\mathcal{S}$; from the Lawvere-Tierney point of view we can directly verify

$$Y^X(C) = \mathrm{Nat}_{\mathbb{C}}(C \times X, Y)$$
$$\Omega(C) = \text{all subfunctors of } C$$

where we have identified C with $\mathbb{C}(-, C) \in \mathcal{S}^{\mathbb{C}^{\mathrm{op}}}$. If $S \in \mathcal{S}^{\mathbb{C}^{\mathrm{op}}}$ then $S \to 1$ is epic iff all $S(C) \neq 0$, but independently of that

$$\mathcal{S}^{\mathbb{C}^{\mathrm{op}}}/S \simeq \mathcal{S}^{(\mathbb{C}/S)^{\mathrm{op}}}$$

where $\mathbb{C}/S$ is a semi-direct product category generalizing the "covering groupoid" of an action S of a group $\mathbb{C}$.

(For another example of the $\mathcal{E}/S$ construction, note that if $\mathcal{X}$ is a space and S a sheaf on it, then $\mathcal{X}/S$ is also a space, namely the corresponding étale space with its local homeomorphism $\mathcal{X}/S \xrightarrow{\pi} \mathcal{X}$.)

I claim that $\mathcal{S}^{\mathbb{C}^{\mathrm{op}}}$ is an étendue iff all maps in $\mathbb{C}$ are monomorphisms. For the last is equivalent to saying that all of the categories $\mathbb{C}/C$ are posets, and $S = \sum_C \mathbb{C}(-, C)$ is a "cover" with

$$\mathbb{C}/S = \sum_C \mathbb{C}/C.$$

To picture these examples, suppose that $\mathbb{C}$ is the additive monoid of non-negative time durations (or more generally abstract time could be replaced by control processes). Then $\mathcal{S}^{\mathbb{C}^{\mathrm{op}}}$ is the category of all *autonomous* (discrete-state, non-necessarily reversible) dynamical systems (or more generally controlled systems) whereas (since $\mathbb{C}$ has cancellation) $\mathbb{C}/S$ is time-considered as a *poset*, with $\mathcal{S}^{(\mathbb{C}/S)^{\mathrm{op}}}$ the category of *non-autonomous* systems. On the other hand, any $\mathcal{S}^{\mathbb{P}^{\mathrm{op}}}$ with $\mathbb{P}$ a poset is spatial with $\mathbb{P}$ a subbase for a freely-generated notion of open set on the set $(\mathbf{Top}/\mathcal{S})(\mathcal{S}, \mathcal{S}^{\mathbb{P}^{\mathrm{op}}})$ of points. (These points p are actually determined by the restriction of p^* along the Yoneda embedding $\mathcal{S}^{\mathbb{P}^{\mathrm{op}}} \longleftarrow \mathbb{P}$, which restrictions are just arbitrary "flat" functors $\mathbb{P} \to \mathcal{S}$.)

For example $\mathcal{S}^{\mathcal{U}^{\text{op}}}$ (for a basis $\mathcal{U}$ on a set $|X|$ as before) is *itself* the topos of sheaves on a larger space $\widehat{X} \supset X$, (whose points are more easily calculated than for general $\mathbb{P}$ since the filteredness of $\mathcal{U}$ makes the above "flatness" condition more understandable).

A general $\mathcal{S}$-topos $\mathcal{X}$ is actually an exact retract $\mathcal{X} \xrightarrow{i} \mathcal{S}^{\mathbb{C}^{\text{op}}}$ for some set $\mathbb{C} \to \mathcal{X}$ of generators. But i_* does *not* in general preserve $\varinjlim$, not even filtered ones (that would be a Nötherian condition) not even finite ones. In particular, the composite $\text{sh}(X) \to \mathcal{S}^{\mathcal{U}^{\text{op}}} \to \mathcal{S}$ (the last being the unique "global sections" functor induced by $X \to 1$) does not preserve epimorphisms. In more detail i^* preserves all $\varinjlim$, hence epimorphisms, thus using i^*, such concepts *in* $\text{sh}(X)$ can be computed from $\mathcal{S}^{\mathcal{U}^{\text{op}}}$ but not *as in* $\mathcal{S}^{\mathcal{U}^{\text{op}}}$: in the latter the objects $U \in \mathcal{U} \hookrightarrow \mathcal{S}^{\mathcal{U}^{\text{op}}}$ are projective, i.e. epimorphisms p are surjective (on each $\text{eval}_{\mathcal{U}}$)

$$\begin{tikzcd} & A \arrow[d, two heads, "p"] \\ U \arrow[ur, dashed, "\exists a"] \arrow[r, "\forall b"'] & B \end{tikzcd}$$

But although $\mathcal{U} \hookrightarrow \text{sh}(X)$ the epimorphisms there are only locally surjective:

for any b, the set of $V \to U$ for which there exists a_V with

$$\begin{tikzcd} V \arrow[r, "a_V"] \arrow[d] & A \arrow[d, two heads, "p"] \\ U \arrow[r, "b"'] & B \end{tikzcd}$$

form a covering of U.

It is partly the above circumstance which makes cohomology possible. The classical example is the question:

Does the complex logarithm exist or not?

to which the well-known answer is "yes and no". If $\mathcal{X}$ is a domain in the complex plane, then

$$\mathbb{C}_{\mathcal{X}} \xrightarrow{\exp} \mathbb{C}^*_{\mathcal{X}}$$

is actually an epimorphism of sheaves. An epimorphism is a map whose image equals its codomain, whereas "$\exists$" is merely the subjective term for the objective concept of image. If b is any never-vanishing continuous function, then $a = \log b$ would mean $\exp a = b$. Since the image = codomain holds, we say that equation is also true in the "internal logic" of $\mathcal{X}$, and with great justice; however, the actual external existence is only on a covering of the domain of b

$$\begin{tikzcd} V \arrow[r, dashed, "\log_V b"] \arrow[d] & \mathbb{C} \arrow[d, two heads, "\exp"] \\ U \arrow[r, "b"'] & \mathbb{C}^*. \end{tikzcd}$$

If Étendue is defined to be the full subcategory of **Top** consisting of locally-spatial toposes, whereas letting a monocat be any category, all of whose maps are mono (e.g. all diffeomorphisms of opens of R^4 *into* other opens of R^4), then we have

$$\begin{tikzcd} & & & \mathbf{top}_{\text{sober}} \arrow[dl] \arrow[dd, hook, "\text{sh}"] \\ & & \text{Étendue} \arrow[dr, hook] & \\ \text{groupoids} \arrow[r, hook] & \text{monocats} \arrow[ur, dashed] \arrow[rr, "\mathcal{S}^{(\ )^{\text{op}}}"'] & & \mathbf{Top}/\mathcal{S}. \end{tikzcd}$$

The inclusion of groupoids into $\mathbf{Top}/\mathcal{S}$ is *up to conjugacy* "full and faithful", whereas up to sobriety, sh is full & faithful (and even 2-full and faithful if we remember the intrinsic ordering on the points of any non-T_1 space). Thus to see the original 1940 "exact sequence" of groups and spaces as living in "spaces" we need only the milder generalization Étendue of **top**. Even that is by no means fully exploited – for example, it seems to me that ergodic theory is largley about "analysis in" a non-spatial étendue. However, Grothendieck wisely realized that the broader generalization $\mathbf{Top}/\mathcal{S}$ contains many examples of geometric relevance, particulary "gros" examples like "all" of algebraic geometry (as opposed to a single variety).

2. Linearized toposes as the base for homological algebra

The linearization $\mathbf{Ab}(\mathcal{X})$ of any topos $\mathcal{X}$, which is just the category of all abelian group objects in $\mathcal{X}$ and homomorphisms thereof, is not only an additive category, but it is an *abelian* category (Image $\xrightarrow{\sim}$ Co-image) and moreover satisfies Grothendieck's "AB5" axiom: filtered (directed) colimits commute with finite limits (kernels). Thus if $\mathcal{E} \xrightarrow{p} \mathcal{B}$ is any morphism of toposes, then its linearization $\mathbf{Ab}(\mathcal{E}) \xrightarrow{p_*} \mathbf{Ab}(\mathcal{B})$ is a right exact functor which has right-derived functors $R^n f_*$. In particular, the global sections functor $\mathcal{B} \xrightarrow{\Gamma} \mathcal{S}$ yields abelian groups $H^n(\mathcal{B}, B) \underset{\text{def}}{=} R^n\Gamma(B)$ for any $B \in \mathbf{Ab}(\mathcal{B})$. In case $\mathcal{B}$ is the topos of sheaves on a space, its cohomology so defined is just that of the space, whereas for the topos $\mathcal{S}^G$ of G-sets, we get the cohomology of the group G with coefficients in any G-*module* B! This unification is even tighter, as sketched below.

Say $\mathcal{E} \xrightarrow{p} \mathcal{B}$ is a "fibration" if $\mathbf{Ab}(p_*)$ takes acyclics to acyclics. Then the spectral sequence of the composed functor $\Gamma_{\mathcal{B}^{\text{op}}} = \Gamma_{\mathcal{E}}$ becomes the Leray spectral sequence

$$H^n(\mathcal{B}, R^m p_* A) \Longrightarrow H^{n+m}(\mathcal{E}, A).$$

An important case is the following:

Let $\mathcal{X}$ be (the sheaves on) a connected, locally connected, semi-locally 1-connected space and let Π_1 be (the permutation representations of) the Poincaré group of $\mathcal{X}$. Then there is a map of toposes

$$\mathcal{X} \to \Pi_1$$

whose inverse image is the inclusion of Π_1-sets as the *locally constant sheaves* (covering spaces). In particular, the self-action of Π_1 is one particular "sheaf" on Π_1, whose inverse image is the "universal" object U. Then the "slice" topos $\widehat{\mathcal{X}} = \mathcal{X}/U$ is the (sheaves on the) universal covering space of $\mathcal{X}$, whereas $\Pi_1/\Pi_1 = 1_{\mathcal{S}}$ (the category of sets) while the resulting (unique) point of Π_1 has the universal covering space as its *pullback* in **Top**

$$
\begin{tikzcd}
\widehat{\mathcal{X}} \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1_{\mathcal{S}} \arrow[d] \\
\mathcal{X} \arrow[r] & \Pi_1.
\end{tikzcd}
$$

Hopf's discovery of the cohomology of groups in the context of topology is now seen as the application of the Leray spectral sequence to this particular exact sequence which crosses over between the two special subcategories of **Top**:

$$
\begin{tikzcd}
\mathbf{top} \arrow[dr, hook, "\mathrm{sh}(-)"] & \\
& \mathbf{Top}/\mathcal{S}. \\
\mathbf{gps} \arrow[ur, hook, "(-)\text{-sets}"'] &
\end{tikzcd}
$$

3. The bornological topos, Banach and Fréchet spaces

Linearization of *gros* toposes leads to functional analysis "rather than" algebraic topology. A surprisingly simple example is the bornological topos $\mathcal{B}$, defined to consists of all (contravariant) functors ${\mathbb{S}_c}^{\mathrm{op}} \xrightarrow{B} \mathcal{S}$ from *countable sets* which preserve finite products in the sense that

$$
\begin{aligned}
B(0) &\xrightarrow{\sim} 1 \\
B(I+J) &\xrightarrow{\sim} B(I) \times B(J).
\end{aligned}
$$

Due to the distributivity of $\mathbb{S}_c$, this simple condition is actually a "sheaf" condition, so that

$$
\mathcal{B} \to \mathcal{S}^{\mathbb{S}_c^{\mathrm{op}}}
$$

is a subtopos, with left adjoint $\overline{(\ )}$ particularly simple to compute

$$
\overline{B}(I) = \qquad \varinjlim_{I_1+\cdots+I_n=I} (B(I_1) \times B(I_2) \times \cdots \times B(I_n))
$$

as the direct limit over all finite partitions of any countable set I. Note that one may as well, for "simplicity" replace $\mathbb{S}_c$ with (*all endomaps of!)* a fixed countable infinite set N. The representable functor $N_c \in \mathcal{B}$ is "codiscrete" whereas the discrete "natural number object" N_d has (N_c, N_d) = only the maps with only a finite number of values; thus $N_d \subseteq N_c$ is infinitely smaller although they have the same *points*. In general $N_c \to X$ is a *bounded* sequence in X and the main idea of a morphism $X \to Y$ in $\mathcal{B}$ is that it takes bounded sequences to bounded sequences, i.e. is "bornological".

If $\mathcal{F}$ is the category of Fréchet spaces (which includes Banach spaces and the even-more-important Fréchet nuclear spaces as almost-disjoint subcategories), then the functor

$$\begin{aligned} \mathcal{F} &\longrightarrow \mathbf{Ab}(\mathcal{B}) \\ F &\longmapsto (I \mapsto \mathrm{bor(I,F)}) \end{aligned}$$

(where $\mathrm{bor}(I, F)$ is the set of all *bounded* functions $I \to F$) is a *full* left exact embedding. The following remark should dispel any lingering belief that functional analysis cannot be considered to be an abelian category. The fact that mono-epis are isos in a topos or abelian category such as $\mathbf{Ab}(\mathcal{B})$ only seems "non-topological" if we neglect that "epi" does not mean just surjective on *points*. Indeed in our example an epi $A \twoheadrightarrow B$ has the property

$$\begin{tikzcd} & A \arrow[d, two heads] \\ N_c \arrow[ur, dashed] \arrow[r] & B \end{tikzcd}$$

that is, bounded sequences in B lift to bounded sequences in A, so that if $A \to B$ is also mono and A, B are Banach spaces, then $A \to B$ is a (linear) homeomorphism.

4. A "petit" topos such as an étendue is essentially a glorified picture of a *particular* "space"

But even in the most elementary geometry the "situation" being studied actually involves points, lines, surfaces, ... and mappings between these, expressing incidence relations, midpoint formations .... The category of all "set-valued continuous functions" X on such a situation is a "gros" topos embodying an idea of *spaces in general* in one of its meatier incarnations. $X(1)$ = the "rational points of X, $X(\text{line})$ = paths in X, $X(\overline{k})$ = k-valued points of X, where $\overline{k}$ is a finite, but not discrete, space of the type studied by Galois, $X(D)$ = tangent vectors of X in many examples. In each of those cases $\overline{A} = l, \overline{k}$, line, D, etc, one actually has

$$X(A) = \mathcal{X}(\overline{A}, X).$$

On the other hand, the set $\mathcal{X}(X, \text{line})$ is the ring of functions on X (or whatever algebraic structures the line has in $\mathcal{X}$), and in many examples there is even an object Λ^n so that (X, Λ^n) = differential n-forms on X. The last really uses the generality of the objects of $\mathcal{X}$, since the "space" Λ^n is non-trivial even though it obviously has only one point! Writing R for the line, we can exploit the fact that $R^X \in \mathcal{X}$ to define

$$\text{Distributions}(X) = \text{Hom}_R(R^X, R) \hookrightarrow R^{R^X}.$$

Of course, all this functional analysis, harmonic analysis, etc. will be "algebraic" if $\mathcal{X}$ = algebraic geometry as explained below, but there are also C^∞ topos $\mathcal{X}$ as studied by Bruno, Bunge, Dubuc, Kock, Lavendhomme, Lawvere, Moerdijk, Reyes, Wraith.

A well-known example of a gros topos is the category of simplicial sets, where since 1950 topology has used the nerve BG which has only one point. In other branches the method is widely under-exploited. The role of gros topos is still not widely recognized.

The construction of a gros topos for geometry usually starts with the *algebra*

$$\mathbb{A} = \mathbb{C}^{\text{op}}$$

of maps between some *basic* generic figures and then finds

$$\mathcal{X} \hookrightarrow \mathcal{S}^{\mathbb{A}}$$

as the sheaves for an appropriate notion on $\mathbb{A}$ of covering. Perhaps the simplest example is $\mathbb{A}$ = the category $\mathbb{S}$ of finite sets, where

$$\mathcal{C} = \mathcal{S}^{\mathbb{S}}$$

is pretty thin on geometry, but does allow to formalize the intuitive notion with which we started Section 1.

$$\begin{aligned}(\mathbf{Top}/\mathcal{S})(\mathcal{X},\mathcal{C}) &= \text{the category of all sheaves on } \mathcal{X} \\ &\simeq \text{“objects in } \mathcal{X}\text{”}\end{aligned}$$

for any $\mathcal{S}$-topos $\mathcal{X}$, so in particular the points of $\mathcal{C}$ “are” just sets

$$\mathrm{Pts}(\mathcal{C}) = (\mathbf{Top}/\mathcal{S})(1_{\mathcal{S}},\mathcal{C}) = \mathcal{S}.$$

Thus $\mathcal{C} \in \mathbf{Top}$ plays the role like that of the complex numbers in **top** of *representing* the “algebra of functions” on any $\mathcal{X}$. If f is any object of $\mathcal{X}$ construed as a sheaf and x is any point of $\mathcal{X}$, then

$$1_{\mathcal{S}} \xrightarrow{x} \mathcal{X} \xrightarrow{f} \mathcal{C}$$

represents a set fx which is the stalk of f at x if $\mathcal{X}$ is spatial; but we had to leave the narrowly spatial Étendue to find $\mathcal{C}$, even if we were only interested in this representing job for spatial $\mathcal{X}$.

Exercise: A point x of $\mathcal{X} = \mathcal{S}^{\mathbf{\Delta}^{\mathrm{op}}}$ (simplicial sets) is exactly any arbitrary linearly ordered set with endpoints. If f is a simplicial set, what is fx?

Thus, like $\mathcal{C}$, many “general” topos are “gros” in the sense that they have a *large* category of points (not only of sheaves). However, this is not the distinguishing feature since, as I explained in my 2005 Theory and Applications of Categories reprint, there is strong reason to consider the category $\mathcal{S}^{\mathbf{\Delta}_1^{\mathrm{op}}}$ of reflexive graphs (= 1-dimensional simplicial sets) as gros even though it has only a *finite* category of points.

Now let k be any good Nötherian ring and let $\mathbb{A}$ be the category of all finitely generated (hence finitely-*presentable*) commutative (associative, unital) k-algebras (or $\mathbb{Z}_2$-graded (anti) commutative algebras if we want “super space” ...). The Yoneda embedding

$$\mathbb{A}^{\mathrm{op}} \xrightarrow{\overline{(\ )}} \mathcal{S}^{\mathbb{A}}$$

will play the role of the (usually very involved to describe) spectrum functor, so we consider only subcategories $\mathcal{X}$ of $\mathcal{S}^{\mathbb{A}}$ which contain $\overline{\mathbb{A}}$ (so-called sub-canonical notions of cover). One central example is the underlying-set-functor

$$\mathbb{A} \xrightarrow{R} \mathcal{S}$$

which is obviously a ring object in $\mathcal{S}^{\mathbb{A}}$ and we straight away consider that $\mathcal{O}_X = \mathcal{X}(X, R)$ for any X. In $\mathcal{X}$

$$R = \overline{k[t]}$$

and for any $A \in \mathbb{A}$ which can be generated by n quantities, there are embeddings

$$\overline{A} \hookrightarrow R^n$$

in $\mathcal{X}$. Moreover

$$\mathcal{O}_{\overline{A}} = A$$

and indeed for any X

$$\mathcal{O}_X = \varprojlim_{x \in \mathbb{A}^{\mathrm{op}}/X} A_x$$

where A_x is the type of the (possibly singular) figure $\overline{A} \xrightarrow{x} X$ of X.

If $D = \overline{k(\varepsilon)}$ where $\varepsilon^2 = 0$, then $D \subset R$, but has only the one rational point $1 \xrightarrow{0} R$ in D; nonetheless X^D is the *tangent bundle* of X, with a map

$$X^R \xrightarrow{(\ )^{\bullet}} X^D$$

from the space of curves in X. If v is any section of $X^D \to X$, then v induces a *derivation* on $\mathcal{O}_X$ (even on R^X).

The "affine" variety such as $y^2 = x^3 + 5$ is simply the functor which assigns to every B in $\mathbb{A}$ the set of all $\langle x, y \rangle \in B^2$ which satisfy the equation; the resulting object X embodies all information about the variety, with no need to first calculate the unnatural conglomerate

$$\text{"pts of } X\text{"} = \varinjlim_{\substack{k \text{ a field} \\ k \in \mathbb{A}}} X(k)$$

or the Zariski opens, or the sheaf of local rings on the petit topos. It is surprising that all three of these ingredients are mandatory in the definition which still passes as "fundamental" in textbooks (even though Grothendieck has himself advocated that it should be abandoned in lectures here at Buffalo in 1973).

As for the Zariski opens themselves, they fit nicely as spaces into our gros toposes:

$$U = \overline{k[t, \frac{1}{t}]}$$

is the functor that assigns to each $A \in \mathbb{A}$ the group of invertible elements of A. The inclusion

$$U \hookrightarrow R$$

is *the* basic Zariski open in that for any $X \xrightarrow{f} R$ (for example any $f \in A$ in case $X = \overline{A}$) the pullback (inverse image)

$$\begin{tikzcd} U_f \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, hook] \\ X \arrow[r, "f"'] & R \end{tikzcd}$$

is a basic open in X. One has $U_{fg} = U_f \cap U_g$, but for a "reasonable" notion of union we need to pass to a subtopos $\mathcal{Z}$ of $\mathcal{S}^{\mathbb{A}}$ known as the gros Zariski topos of k. Actually,

$$\mathbb{A}^{\mathrm{op}} \hookrightarrow \mathcal{F} \hookrightarrow \mathcal{E} \hookrightarrow \mathcal{Z} \hookrightarrow \mathcal{G} \hookrightarrow \mathcal{S}^{\mathbb{A}}$$

are all four significant toposes for algebraic geometry; first we consider the simplest one $\mathcal{G}$ which consists of all *product-preserving* functors. Just as with the countable sets in the bornological example, it is the unusual *distributivity* of $\mathbb{A}^{\mathrm{op}}$ which makes $\mathcal{G}$ so defined a topos; *un*like the bornological example $\mathcal{G} = \mathcal{S}^{\mathbb{A}_0}$ is itself a full functor category, where $\mathbb{A}_0 \subset \mathbb{A}$ is the category of those k-algebras satisfying

$$\begin{aligned} 0 &\neq 1 \\ f^2 = f &\Rightarrow f = 0 \text{ or } f = 1. \end{aligned}$$

The colimits in $\mathcal{G}$ differ from those in $\mathcal{S}^{\mathbb{A}}$ to the extent that $\overline{A \times B} = \overline{A} + \overline{B}$ is a coproduct in $\mathcal{G}$. The somewhat more subtle $\mathcal{Z}$ is defined to consist of those X in $\mathcal{G}$ which transform

$$U_0 \cup U_1 \hookrightarrow R$$

into an isomorphism, where $U_0 = U$ and U_1 is its translate from 0 to 1. The last generates a good notion of covering so that $\mathcal{Z}$ too is a topos, now with even more colimits agreeing with those in $\mathbb{A}^{\mathrm{op}}$. In fact in $\mathcal{Z}$ we have

$$U_{f+g} \subseteq U_f \vee U_g$$

for any two functions on any space X, or equivalently that R itself is now a *local* ring in the sense of the internal logic of $\mathcal{Z}$. The image (in the sense of $\mathcal{Z}$) of the characteristic map of the basic open

$$\begin{tikzcd} U \arrow[rr] \arrow[d, hook] \arrow[drr, phantom, "\lrcorner", very near start] & & 1 \arrow[d, "\text{true}"] \\ R \arrow[r, two heads] & \Omega_b \arrow[r] & \Omega_{\mathcal{Z}} \end{tikzcd}$$

is not only a meet-semilattice classifying the basic opens, but there is also a distributive (Joyal) lattice between Ω_b and $\Omega_{\mathcal{Z}}$ which classifies the more general opens. In order to classify locally-closed sub-"schemes", consider the image in $\mathcal{Z}$ of $K \to \Omega_{\mathcal{Z}}$ where K is the Dedekind-Krull functor $K(A) =$ all ideals of A, with the co-variant functoriality $\alpha \mapsto B{\cdot}\varphi(\alpha)$ for any $A \xrightarrow{\varphi} B$ in $\mathbb{A}$.

The étale topos $\mathcal{E}$ defines the further restriction that X transforms a map p into an epimorphism if p satisfies that

$$\begin{tikzcd} E^D \arrow[r, "p^D"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B^D \arrow[d] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

is a pullback (i.e. p satisfies the hypothesis of the implicit function theorem) and also p is surjective on points:

$$\begin{tikzcd} \overline{k'} \arrow[r, dashed, "e"] \arrow[d] & E \arrow[d, "p"] \\ \overline{k} \arrow[r, "b"'] & B. \end{tikzcd}$$

"for all fields k in $\mathbb{A}$ and all $b \in B(k)$ there is a field extension k' in $\mathbb{A}$ and $e \in E(k')$ such that $pe = b$ modulo the extension". $\mathcal{F}$ is defined similarly but using "faithfully flat" p. The Grassmann manifolds $\mathrm{mono}(W, V)/\mathrm{aut}(W)$ for R-modules W, V are definable in all these toposes, but the meaning of the quotient space involved in general depends on which. Group objects in these toposes have very remarkable properties.

The 1989 version of this paper was created in a great hurry! Bill told me[1] on a weekend that he was planning to hand out a paper to the participants of the seminar on Monday; but the paper did not exist yet. So he proceeded to write the paper by hand, and I typed the pages as they were produced, on our typewriter. Then I drew the arrows, the Greek letters, the displays. Bill read and changed some page that I typed again. Then we went to Kinko to have 100 papers photocopied and collated. And on Monday he distributed them. (Very fortunately, he brought one back home or we would never have known about this paper.) A decade later he happened to see it, and he corrected it extensively. It looked pretty miserable by then.

But now in 2025, our friend Francisco Marmolejo offered to reformat it. [He has our same wish to get Bill's unpublished papers out in the open, for some scientist to read, or some student to study them, or for somebody to find a special nugget once in a while, to help the science of the future.]

[1] His companion Fatima of 57 years.

We very much appreciate Francisco's keen eye for detail, his patience, his generous time, his knowledge, and his persistence. Thanks Quico for your transformation of the old version into a very beautiful, easily readable paper!

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