Basic dynamics via category theory

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In fond memory of Walter Noll

Abstract. In the 1950's it was clear that abstract conceptual mathematics was being re-organized by category theory. In the same period Walter Noll was undertaking the application of abstract conceptual mathematics to Mechanics. These two trends were not without interaction, especially in the environment provided by Clifford Truesdell. I offer here some applications of category theory to the preliminary conceptual organization of basic dynamics, arriving at a notion of force that unites the cases of a system of particles and the case of a continuous body, revealing in particular a special role for internal forces.

An important tool is adjointness introduced by Kan (1958). An appropriate setting is a Cartesian closed category (Eilenberg and Kelly 1966), which is a category usually embodying a form of smoothness or cohesion, but especially admitting function spaces characterized by the basic transformation rule used by Bernoulli, Volterra, and Hurewicz. From the modern approach (Bunge, Kock, Lavendhomme, Moerdijk & Reyes) to Differential Topology and Geometry that makes use of nilpotent quantities, I use here mainly its general functorial aspect in terms of prolongation operators.

There have been in the 20th century useful theories of 'dynamical systems' that take place in a suitable category, but typically involve only an additive monoid of time acting on a space of states. Of course, a state is not just a configuration, but is rather a state of motion (as in the works of Fibonacci and Galileo), and the term 'dynamical' suggests some role for force. I will include explicitly these two aspects (states of motion and a role for force) into a refinement of the usual algebraic formulation.

I use the natural map $X \to C$, where X is the tangent bundle of C, to represent the relation between states and configurations. A third feature that I want to incorporate in the algebra is Newton's description of a force as something that transforms a straight-line law of motion into another law of motion. A law of motion can be characterized as a section of the restriction $X' \to X$, where X' is the second tangent bundle of C. In that context, a force law can be defined to be an endomorphism f of X' which fixes the restriction map to X. Obviously, these forces can be composed; the resulting monoid expresses the successive applicability of force laws to laws of motion.

How can this composition be made compatible with the traditional teaching that forces live in a vector space and compose by vector addition? The restriction to infinitesimals clarifies this. The Lie algebra of the monoid embodies, in its addition, the composition of the infinitesimal forces; it is indeed commutative by the Eckmann-Hilton lemma. The history of Lie algebra raises the question how, in engineering situations, the orders in which various forces are applied may be non-commutative in their effects.

(A straight-line law is one that commutes with the natural action of speed-ups on $X' \to X$, where by speed-up I mean a natural endomorphism of X as a functor; such induces also an action on X'. Question: are force laws in fact determined by their restriction to their action on straight-line laws of motion?)

The structure described above can be intimately related to the nature of the configuration space C as (part of) a function space E^B where B is a body (continuous or not) and E is ordinary space. Using the convenient representation (from Synthetic Differential Geometry) of the tangent bundles as function spaces on spaces D of infinitesimals ($X = C^{D_1}$, $X' = C^{D_2}$), we can distinguish internal and external force laws. Here an external force is simply induced up from a given law on E itself, whereas the internal force may be co-induced into E by a simple constitutive relation on B, namely an endomorphism of $B \times D_2$ that fixes the subspace $B \times D_1$.

There are two distinct quantitative viewpoints, usually attributed to Euler and Lagrange, concerning a fluid body B; these may be distinguished by the choice of configuration concept: either measures on E, or metrics on B, which may be viewed as the reflection in E or in B (via the interaction between the two) of a mass measure on B or a metric on E. (Of course, a flow of point placements will induce a flow in each of these.) Whether the measures remain absolutely continuous during a motion is sometimes a basic question.

In the category of bodies B equipped with a force law, a morphism $T \to X$ may in particular be just a solution of the differential equation implicit in X. That would be consistent with the proposal of Einstein to construe time T itself as nothing but a given clock.

1 Author's additional note (June 28, 2018)

I would like to rewrite the paper, to bring out more clearly some developments that are new, and can conveniently be expressed in categorical language. The abstract and the introduction will contain essentially the same information as below.

An important part of the categorical language is the theory of Cartesian-closed categories, which characterizes function spaces, using the basic transformation of Bernoulli, Volterra, and Hurewicz: there is a natural bijection between maps $X \to Y^A$ and maps $A \times X \to Y$.

For several decades it has been customary in mathematics to study topological dynamics, involving a chosen space of states and a chosen monoid object, together with an action of the monoid on the states. Rather than the category of topological spaces, another category may be appropriate, such as that of Borel spaces, or of smooth C^∞-spaces.

In the smooth category, one can also consider vector fields as an infinitesimal version of the monoid actions. In general, states are interpreted as states of becoming (involving perhaps velocities or histories) rather than states of being (also known as configurations); that is why laws of motion are so frequently second order differential equations.

In particular, I interpret configurations as placements of a body B in a space E. (The space E may have mainly geometrical significance, or may involve other parameters.)

Only part of the placement space E^B may be relevant. I want to incorporate in such algebra the role of laws of force and their relation to laws of motion. To my knowledge, the particular interpretation given here for Newton's famous remark has not been published before. One result is that the laws of force act on laws of motion and hence form a monoid in their own right; and the internal force in a body B receives a description that seems to be new too.

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