Cantor's *lauter Einsen* in the light of Galois and Grothendieck

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Abstract. For 20th century mathematicians, the role of Cantor's sets has been that of the ideally featureless canvases on which all needed algebraic and geometrical structures can be painted. (Certain passages in Cantor's writings refer to this role.) Clearly, the resulting contradiction, the points of such sets are distinct yet indistinguishable', should not lead to inconsistency. Indeed, the productive nature of this dialectic is made explicit by a method fruitful in other parts of mathematics.[1]This role of Cantor's theory is compared with the role of Galois theory in algebraic geometry.

It is often said that sets have something to do with the foundations of mathematics. Why then does the pursuit of mathematical subjects, such as algebra, geometry, analysis, combinatorics, et cetera, refer so infrequently to the results of professional set theorists? A more fundamental question is 'What are sets?' For professional set theorists, the default answer to that question has been the cumulative hierarchy. But even the recognizably set-theoretic questions that occasionally arise in mathematics, such as choice, the continuum hypothesis, measurable cardinals, et cetera, have direct mathematical meaning independent of that hierarchy.

In advanced calculus (if not earlier) students of mathematical science learn the utility of explicitly considering **subsets** of a universe such as the plane or 3-space. Subsets of

[1]See 'Adjointness in Foundations', Dialectics 1969

a given universe may be included in one another, and figures (for example points) may belong to such subsets. There is certainly no preferred inclusion of one universe of discourse into another (on the contrary, there are for example many plane-shaped figures in 3-space). The explicit consideration of a set that parameterizes the subsets of a given universe, as a further universe in its own right, is also very relevant and useful. However, the notion that the elements of these further sets can be unambiguously 'accumulated' results from idealist philosophy and introduces unnatural complications, obscuring not only the role of sets as carriers of mathematical structures but even Cantorian issues of set theory itself.

Of course, the specifications of subsets of a given universe often proceed in terms of properties that elements might have. This led the idealist philosopher Frege to the speculation that all concepts are properties. Abandoning the logical principle that properties are defined on a definite universe of discourse, his properties could only be defined on 'everything'. But actually the meaning of properties is only relative to a structure that the universe of discourse carries. Thus beyond generalities about inclusions, Frege's 'everything' required some specific structure, so he was pleased when Peano made explicit the following: Among all the sets (= subsets of the grand universe) there are the minimal or atomic ones A having the metaproperty that

$$[B \subseteq A] \Longrightarrow B = 0 \text{ or } B = A;$$

let us assume that testing with inclusions of atoms is adequate for determining general inclusions (this extensionality does not involve singleton or epsilon). Moreover, let us imagine that every set X has been assigned a name that is such an atom, denoted by $\{X\}$. In terms of this, Peano's other contribution, the epsilon, is defined by

$$X \in A \text{ iff } \{X\} \subseteq A.$$

Cantor protested vehemently against all that. Zermelo, however, seems to have sided with Frege against Cantor, and von Neumann followed Zermelo, leading directly to the cumulative hierarchy and parting with the rest of mathematics. The prestige of von Neumann, together with Goedel's application, led to the acceptance of that hierarchy as the default determination of 'set theory'.

The line of set theory starting with Cantor and Dedekind, and largely followed by

Schroeder, Hausdorff, and indeed most of 20th century mathematics, was instead derived by conceptualizing the developments in the spirit of Riemann, Volterra, Dedekind, and Hausdorff.

The sets actually occurring in geometry and analysis bear rich structure of cohesion and variation (for example, a variable circle can have various centers, and two specific circles may be near as measured by other circles).

The younger Cantor's brilliant idea was that one could abstract away the structure to obtain sets of '*lauter Einsen*' or bare points devoid of structure; such sets would bring into sharp relief their cardinalities, permitting one to compare them after the manner of Steiner and, in particular, to prove by the 'diagonal argument' that the power sets are always 'bigger'.

But of course this abstraction was not an end in itself, but rather one aspect of a richly dialectical conceptual process. The abstract sets serve as bearers for possible structures that could be thus made mathematically more explicit; that application, which began immediately with Cantor and Dedekind, has been a key feature of conceptual mathematics ever since.

An important further development of the dialectic is that often algebraic structures of interest are not only abstract, but also modeled in a cohesive background, for example, Lie groups in a smooth background, or Banach spaces in a bornological background. Perhaps it was only via topos theory that the powerful analogies between different levels of cohesion and variation, including the zeroth level of total abstractness, could be made explicit; but these analogies have been long present in practice. Part of the analogy is that all such levels can be construed, like the abstract zeroth case, as toposes, with the resulting rich development of internal constructions. A powerful method in analysis, geometry, and logic, is to creatively 'lift' constructions, that are first understood for the background of abstract Cantorian sets, to more general toposes of cohesion and variation.

1.

The virtue of considering a base topos of objects that are highly discrete, yet that are not totally abstract, was already implicit in the work of Galois. The 'set' of solutions of an equation has after all a bit of cohesion and variation if we consider that the coefficients of the polynomial and rational functions may come from a base field (like the rational numbers) that is not algebraically closed. The points that are the solutions of a separable equation may come from algebraic extension fields, and considering the set of such solutions by themselves yields a space that is zero-dimensional (in the spirit of Leibniz) but not totally discrete.

Such zero-dimensional algebraic spaces constitute a topos $\mathcal{G}_0$ in their own right, as pointed out by Barr; it is the essential object of study for classical Galois theory. On the other hand, the category of algebraically-defined spaces of all finite and infinite-dimensions is seen also to be a topos $\mathcal{G}_1$ on the basis of work of Grothendieck 1960, Gabriel 1966, and Grothendieck 1973. While there is a default grounding of $\mathcal{G}_1$ in abstract sets by considering all prime ideals together, that process does not even preserve Cartesian products. The grounding $\mathcal{G}_1 \to \mathcal{G}_0$ avoids such discontinuities, yet expresses well the dialectic cohesion/non cohesion involved in the calculations of algebraic geometry.

Of course, the calculations that had been made using the von Neumann hierarchy have led to some striking results. The essential content of some of the most striking involves (as pointed out in 1970 by Myles Tierney) Boolean extensions $\mathcal{B}_0$ included in $\mathcal{B}_1$ of toposes that have right adjoint groundings $\mathcal{B}_1 \to \mathcal{B}_0$. What the 'independence results' actually reveal is the inadequacy of the usual description to express Cantor's idea of abstract sets. His insight was that the infinite sets A and B, devoid of cohesion and variation, would satisfy the dichotomy $A \leq B$ or $A \geq 2^B$. The work of Goedel, Dana Scott, et al indirectly shows that this continuum hypothesis is actually true. However, nobody has as yet drawn that conclusion in a rigorous way, primarily because set theory lacks an account of relative cohesion and variation. I propose that such an account can be devised by relaxing the requirement that comparison functors between toposes be left-adjoints.

2.

A second proposal concerns the treatment of 'size' questions in the category of categories (and hence in the foundations of mathematics). [Since we want to concentrate on first order axioms, 'the' should be replaced by 'a' generic model for the generic theory.] Of course, categories are trivially identifiable with finite diagrams in a set-like category. For either sort of universe of discourse, Frege was of course right in saying that sets (or categories) may have a certain property, and that an important contradiction concerns 'collecting' or representing such by an actual category $\mathcal{C}$ (i.e. by an object in the universe). For many properties the appropriate mode of representing is not via the rigid epsilon, but rather via categorical equivalence. For example, the idealization known as the axiom of infinity, in its objective form, states that for the property of being finite and discrete, there exists a category $\mathcal{S}$ of finite sets such that for every A satisfying the property, there is an object a in $\mathcal{S}$ and an isomorphism $\mathcal{S}(1, a) = A$.

Because categories themselves have become objects of study, there is the need to extend that idealization to recognize an actual category $\mathcal{U}$ of small sets that will support all the constructions arising within ordinary geometry and analysis, finite and infinite dimensional. The 1945 work by Mac Lane, was too broad in the sense that it followed his teacher Bernays by identifying small with representable, and yet too narrow in the sense that it did not satisfy the fundamental principle that for any two actual categories, their functor category is also actual, as wisely required by Grothendieck in his 1958 Tohoku paper. Mac Lane's work 15 years later still had the drawback of condemning some natural functor categories, like $\mathcal{U}^{\mathcal{U}}$, as 'illegitimate' existence; that usage regrettably became the default; that work of Mac Lane had the great virtue that under the concept of a 'locally small' category $\mathcal{C}$, it popularized the method of enriched categories relative to a monoidal closed category $\mathcal{U}$, i.e. categories for which $\mathcal{C}(x, y)$ is isomorphic to an object of $\mathcal{U}$ for any two objects x, y of $\mathcal{C}$.

Grothendieck, in addressing the definition of smallness, in effect identified $\mathcal{U}$ with the topos of sets smaller than the first inaccessible cardinal. (Then of course we need a further topos $\mathcal{V}$ having objects equivalent to $\mathcal{U}^{\mathcal{U}}$ and similar 'large categories'.) But instead of pursuing the speculation of eternal iteration, let us ask if there is a reasonable direct constraint on how large 'small' can be. Indeed, such a constraint had been discovered (in contra positive form) by Banach, Tarski, and Ulam around 1930, and recognized as such before 1960 by workers in the area of rings of continuous functions. The fundamental contradiction of mathematics, measuring space by means of variable quantities, led to the notion of a realcompact space as one for which that process works fully. All known metrizable spaces have that property, and conversely, one can define a 'small' cardinal X as one that is realcompact, i.e. as one for which every real homomorphism $\mathbb{R}^X \to \mathbb{R}$ comes from a point of

X. In 1960 Isbell noted a more abstract 'set-theoretical' equivalent, replacing $\mathbb{R}$ by natural numbers $\mathbb{N}$ and 'homomorphism' by naturality with respect to all elements of $\mathbb{N}^{\mathbb{N}}$ considered as operations. In case there are sets X bigger than that, it is due to the fact that $\mathbb{N}^{\mathbb{N}}$ (= 'the reals' in the set-theorists terminology) is too small to sort out the actual evaluations on the observations $X \to \mathbb{N}$. (A historically attractive alternative formulation would replace $\mathbb{N}$ by the Euclidean plane and $\mathbb{N}^{\mathbb{N}}$ by the monoid of operations generated by the conic sections).

Since categorical practice seems to require that small sets be represented by an actual category $\mathcal{U}$, and since postulating the representability $\mathcal{S}$ of finiteness did not seem to lead to a contradiction, we are led to postulate also the existence of the above maximal geometrically-natural choice of $\mathcal{U}$. This postulate is equivalent to the mildest of all the 'large cardinal' assumptions commonly made by set-theorists, so much is known about it.

From the point of view of applications to mathematics, the most important thing known about small sets is that they are closed under exponentiation and thus form a topos. The calculation that proves that fact can be interpreted in the much simpler setting of any elementary topos (or even type theory). Consider a given dualizing object R and the monoid $M = R^R$. There is a natural map

$$X \to \mathrm{Hom}_M(R^X, R)$$

from any object X into the part of the double dual consisting of those functionals compatible with the action of M. Those functionals are to be thought of as 'ghost elements'. Consider the subcategory of 'M-small' objects X for which that natural map is an isomorphism. That **subcategory** should again be a topos (a further requirement would be an actual **category object** that represents it). This calculation can be explained in even more categorical terms by noting that the X in question are natural fixed points of the functor

$$TX = \mathrm{Hom}_M(R^X, R)$$

and that T is actually a monad. Natural fixed points are special Eilenberg-Moore algebras for T, for which the algebra structure is unique. For any algebra A for T, and for any object X, A^X has again a determined T-Algebra structure by very general principles; thus the point is that if A is actually fixed (as is typically $A = \Omega$) then A^X is also fixed if X is. The natural algebra structure on A^X provides, for each ghost element of A^X, a canonical choice of an actual element; that idempotent process (existing in any third dual) should be the identity when X is also fixed.

3.

The above two proposals should be compatible, as suggested by old results of Levy. However, as shown by Dana Scott, the existence of $\mathcal{U}$ seems incompatible with the method L of achieving extreme constancy that was used by Goedel. Although the extreme constancy as a concept has nothing to do with iteration, extreme iteration is used to 'generate' L, thus endowing its sets with very involved structure. However, passing to the category **cat**(L) of sets and maps definable in L, there results a topos whose objects are structureless enough to satisfy Cantor's generalized continuum hypothesis. Thus a method milder than Goedel's is needed to describe a model of Cantor's *lauter Einsen* that satisfies both the generalized continuum hypothesis and the representability $\mathcal{U}$ of smallness. I propose to focus on the category structure to free such definitions from the encumbrances inherited from the rigid hierarchies.

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