# Category Theory and everyday human activities

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Everyday human activities such as
  building a house on a slope;
  laying a network of telephone conduits;
  navigating the solar system;
require plans that can work.

Planning any such undertaking requires the development of

Thinking about Space.

Each development involves many steps of thought and also many related geometrical constructions on spaces. Because of the necessary multi-step nature of this thinking about space, uniquely mathematical measures must be taken to make it reliable. Explicit principles of thinking (logic) and explicit principles of space (geometry) are a needed part of a guaranteed reliability.

The great advance made by category theory, invented 60 years ago by Eilenberg and Mac Lane, permitted mathematicians to make the principles of logic and geometry explicit. This was accomplished by discovering the common form of logic and geometry; therefore the principles of the relation between the two have also become explicit. In an important sense this has solved a problem opened 2000 years ago by Aristotle with his initial inroads into making the categories of concepts.

In the 21st century, this solution is applicable not only to plane geometry and to medieval syllogisms, but also to infinite-dimensional spaces of transformations, to 'spaces' of data, and to other conceptual tools that are applied thousands of times a day. The form of the principles of both logic and geometry was discovered by categorists to rest on 'naturality' of the transformations between spaces and the transformations between thoughts. Naturality applies also to the transformations between Thinking and Space; that relation itself thus became subject to reliable elaboration and development on the basis of the same unifying principle.

What, more precisely, is this 'naturality'? It does not refer to something spontaneously arising in Nature (although it may be important in theories of physics), but rather to the natural expectations of careful Thinking.

Among all possible performances in the presence of a plan, we may need to single out those that are appropriate executions of the plan. For example, a musical performance produces a map from a time interval into a space of sounds (possible frequencies and amplitudes), but a recital involves both a real performance (in real time and sound) and also a symbolic 'performance' by the composer in a symbolic time interval and a symbolic space of sound. (For centuries, composers have written the symbolic time horizontally and the symbolic sound space vertically, though we do not show that here.) Thus the abstract idea $x$, that symbolic Notation can be interpreted into Reality. Reality has many relevant instances, in particular, is common to both

$$\begin{aligned} \mathcal{T} &= \text{measuring of a time interval, and} \\ \mathcal{S} &= \text{specification of sound meaning.} \end{aligned}$$

(For example, we might consider the $\mathcal{S}$ implicit in the work of Guido d'Arezzo.)[1] Then a symbolic performance by a composer, together with a real performance by an artist constitutes a natural recital $\rho : \mathcal{T} \to \mathcal{S}$ if the naturality equation holds

$$\rho(\mathfrak{R})\mathcal{T}(x) = \mathcal{S}(x)\rho(\mathfrak{N}).$$

In general, there is more than just the one aspect $x$ of structure common to both, source and target of a natural process, but the equation is the same for each. A helpful geometric view of such equations, is provided by diagrams like

$$\begin{array}{ccccc} \mathcal{T}(\mathfrak{N}) & \xrightarrow{\rho(\mathfrak{N})} & \mathcal{S}(\mathfrak{N}) & & \mathfrak{N} \\ {\scriptstyle \mathcal{T}(x)}\downarrow & & \downarrow{\scriptstyle \mathcal{T}(x)} & \text{for all} & {\scriptstyle x}\downarrow \\ \mathcal{T}(\mathfrak{R}) & \xrightarrow[\rho(\mathfrak{R})]{} & \mathcal{S}(\mathfrak{R}) & & \mathfrak{R} \end{array}$$

read: $\mathfrak{N}$ = symbolic $\qquad \mathcal{T}(x) = eg$ pupil's metronome

Both intuitively and mathematically, we may say in general that in

$$\mathbf{SPACE}\circlearrowleft \;\rightleftarrows\; \mathbf{THINKING}\circlearrowleft$$

all four arrows are natural!

[1] Guido d'Arezzo wrote 'The Fundamentals of Music', a very important medieval treatise; it is true, because he is widely credited with developing the modern Western musical notation system. Guido d'Arezzo, a medieval music theorist, lived in the 11th century. (Wikipedia)