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Induction is Idempotent Recursion in a Bicategory

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The essentially bicategorical axiomatization of induction by Vaughan Pratt suggests an important way in which induction is related to recursion, and equationally exploits an important strengthening of the central universal property.

Notation

The term 'semi-ring' strikes most mathematicians as describing a semi-legitimate subject, and indeed the concept expresses less than half of what the positive results about them actually prove, namely, a semi-ring is only (the endomorphisms of) a single object in a LINEAR CATEGORY which probably has several more objects that are of direct related interest in any given application (square matrices operate on rectangular matrices that aren't square, for example $(k \times n)$ and $(n \times j)$ matrices may be inputs and outputs for $(n \times n)$ processes, etc.). Hence the radical programmatic proposal: drop the term from mathematics and concentrate on the more useful study of linear categories, naturally paying much special attention from time to time to linear categories which have only a few objects. Note that the property of having exactly one object is extremely unstable, for if we discover in our semi-ring an idempotent of particular interest, it would be wise to split it because, without appreciably changing the mathematical content, that additional object and the diagrams (with precise domains and codomains) which go with it do improve considerably our conceptual overview and computational flexibility with the problems at hand. The opposite process, namely given a finite number of objects in a linear category, to consider only endomorphisms of their (bi-)product, is often a hollow “unification”, since at the cost of using much larger matrices, one has also obfuscated the clarity of those diagrams which show instantly the distinctions between inputs, outputs, feedbacks, etc.. Of course, commutative semi-rings play additional very special roles, which is why Steve Schanuel and I gave them the positive name Rig (a ring without required negatives).

A 2-category is a 2-category (even a 2-CAT category)

More precisely, a 2-linear category is naturally a bicategory. A linear category is one in which finite coproducts exist and are canonically the same as finite products (Mac Lane 1950); the center of such a category is always a rig, which will contain the rig 2 (in which $1 + 1 = 1$) iff the composite $X \to (X \times X) \to X$ (codiagonal following diagonal) is the identity on X. Any linear category whose center contains a rig K will be naturally enriched in the standard closed monoidal K-linear category of K-modules, which in case $X = 2$ is also known as the category of semilattices.

But now the canonical functor from K-modules to posets (preserving underlying sets, with x (weakly) less than y defined to mean, as in the “specific order” of potential theory, that there exists $z[x + z = y]$ which in case $K = 2$ of course becomes equational because some z proves it iff y itself does) is actually a monoidal functor, because there are natural order-preserving maps

$$1 \longrightarrow K$$

$$(A \times B) \longrightarrow AB$$

the latter denoting the tensor product of K-modules A, B, but considered merely as a poset. For if $a + x = a'$ and $b + y = b'$, then $ay + xb + xy$ proves that ab precedes $a'b'$ in the tensor product.

Since a monoidal functor transports one enrichment of a category into another enrichment, and since posets are a cartesian closed subcategory of the category of categories, I have thus shown that any K-linear category a has an underlying bicategory. Of course, if the rig R is actually a ring (i.e. has negatives), then all objects in typical hom-categories $a(V, W)$ are uniquely isomorphic, and indeed for many K there is much collapsing. But in case $K = 2$

if x and y are comparable in both directions, then $x = y$, so that the uniqueness of adjoint arrows means strictly their uniqueness. Indeed, the possibility of relating arrows in a 2-linear category by adjointness is one of the main reasons for making explicit that such categories are in particular bicategories (of course we will have to take care to distinguish categorical adjointness from the linear-algebra adjointness which might now be taking place in the same category.)

The pleasant feature emphasized by Vaughan Pratt, namely that categorical adjointness can be fully expressed by semi-lattice equations, of course continues to hold in all these 2-linear categories.

Induction versus recursion

Both of these involve certain special adjoints/monads. The traditional distinction, that induction concerns mere statements, whereas recursion supposedly constructs ideal entities to which the statements might refer, is implicit in Vaughan's description: The Kleene-Pratt operation star is a monad which is idempotent, whereas non-idempotent monads are typical for recursion. A basic kind of recursion is embodied in the monad, on the category of graph objects in a given lextensive category, which constructs the free internal category (of "paths"). (Here by graph I mean any pair of maps "source/target", that is, a directed multigraph, so that a graph with one node is the same as a plain object X, in which case the path-category TX over X is the free monoid object.) Since I said lextensive, instead of merely lex, to exclude the case of a lex poset (whereon all monads would be idempotent), the unit and multiplication for the monad T are not expected to be isomorphisms, so that T is not naturally idempotent in that sense.

But other than the above idempotent/non idempotent distinction, there is another important quality of certain monads, somewhat known in the nonidempotent (recursion) case, but also, as Vaughan is implicitly pointing out, also important in the idempotent (induction) case. Consider again a given object (or more generally graph) X in a lextensive category E. There is a category of X-actions in E, wherein an object is an arbitrary object A of E equipped with an arbitrary map $(X \times A) \to A$, with the obvious X-equivariant morphisms. The left adjoint, if it exists, to the forgetful functor $(E^X) \to E$, would give free X-actions on any object Z of zeroes. That adjoint is very important in recursion, but may not exist even when free categories (e.g. free monoids) do. The free X-action on Z should be just $(T(X) \times Z)$, but if higher-order (=cartesian closed) structure is lacking in E, there may not be enough monoids, so that the requirement of universality with respect to all X-actions may be strictly stronger. This contradiction is quite objective; it is the source of many of the crucial properties studied in topological dynamics, where the category E of compact topological spaces is lextensive, but not closed, and the actions of the word monoid $T(2)$ contain many subtleties.

There are of course many non-cartesian (i.e. non lex) situations where similar constructions arise. For V-enriched categories over closed monoids V, there are tensor algebras and also the least-cost distance metric on a weighted graph. In my 1973 paper [**?**] I say that the universal property of the free V-category on a V-graph expresses the essence of recursion. That should be supplemented with a mention of free actions, although their existence follows from the higher-order structure in the case of the world of V-categories. But as Vaughan's construction shows, recursion also makes sense in categories which are merely enriched in V-cat (without being equal to it).

That actions (presheaves, modules...) are involved in Vaughan's construction, not only monads (categories, transitive elements...) is illustrated when he speaks of Q preserving P, $PQ = P$: P can be a rectangular matrix even though Q must be square.

Residuation and recursion in bicategories

In a bicategory a, composition

$$a(A, B) \times a(B, C) \longrightarrow a(A, C)$$

is sometimes denoted with a tensor symbol, correctly suggesting that there may be right adjoints

$$a(A, B) \longleftarrow a(B, C)^{op} \times a(A, C)$$

$$a(B, C) \longleftarrow a(A, C) \times a(A, B)^{op}$$

which could reasonably be denoted by Hom, as in Cartan-Eilenberg. These are the same as those which in the posetal case are sometimes called residuation and denoted by slashes.

Also in a bicategory there are at each object A not only the category of endomaps $a(A, A)$ but also the category $Mon\ a(A, A)$ of monads with its forgetful functor to $a(A, A)$. If the latter has an adjoint, assigning, to each endo arrow P of A, the free monad P^* it is of course unique. It will be idempotent $P^{**} = P^*$, in case $a(A, A)$ is a poset, as in the case of "induction", where for example a is enriched in posets because it is 2-linear, as described above. In other cases, such as V-graphs with vertices A, the free category or recursion $monad(\ )^*$ is rarely idempotent, although it fits in the same 2-categorical definition.

QUESTION: If induction is the subjective reflection of objective recursion, can particular instances of the two be explicitly linked by something like a support functor or a Schanuel dimension [3]?

It seems important that the Kleene induction as broadened by Pratt also involves free actions, not just free monads. If $Q : A' \to A$ and $X : A \to A$, then the idea that X preserve Q, $XQ = Q$ could be broadened to giving an action $XQ < X$ of X on Q, which by functorality of composition would lead to $X^2Q < XQ < Q$ etc., hence $X^*Q < Q$, i.e. an action of the free monad. Similarly, for the actions of X on arrows $A \to A''$.

In connection with the study of representing PDEs in a smooth topos, one needs to iterate objects X in a bicategory (namely that of distributions between toposes) in which X has a right adjoint already, that's very special since, although due to the residuation, composing $X(\,)$ with X already has a right adjoint functor $Hom(X, -)$, that functor only rarely has itself the form $1/X(\,)$ of composing with another object $1/X$. This leads to additional derived rules of inference.

But where do the residuated bicategories themselves come from? In examples, even the composition is typically justified by using co-completeness of the categories $a(A, B)$, while the Homs (or residuations) are justified by their completeness. Vaughan's treatment of induction is thus in accordance with a categorical approach to the foundation of many subjects, whereby one replaces completeness by the explicit postulation of the key adjoints which express the transformation rules essential to the subject.

References

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