Chapter 13
Euler's Continuum Functorially Vindicated

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Contrary to common opinion, the question "what is the continuum?" does not have a final answer (Bell, 2005), the immortal work of Dedekind notwithstanding. There is a deeper answer implicit in an observation of Euler. Although it has often been dismissed as naive, we can use the precision of the theory of categories to reveal Euler's observation to be an appropriate foundation for smooth and analytic geometry and analysis.

Euler observed that real numbers are ratios of infinitesimals. This is not only true but an effective definition of the smooth real continuum; the properties of the smooth continuum, including its canonical map to the less-refined Dedekind continuum, can be expressed in terms of such ratios.

Some of the features of categorical precision are the following. Maps (functions, transformations, etc.) have both definite domains and definite codomains, with operations like restriction to subdomains or expansion of codomains being effected by composition with suitable inclusion maps; such operations change the properties of the map. Equality of maps $A \to B$ can be tested by composing with elements $E \to A$ where E belongs to a class of element-types chosen in a way appropriate to the particular category; the limitation to a bare point $E = 1$ is typically not appropriate (except for the special abstract constant sets that Cantor extracted from mathematics for his particular purposes). It is possible and preferable to assume for basic work in geometry and analysis that the ambient categories are cartesian closed, as was already taken for granted by Euler's teachers: for two spaces X, Y there is a well-determined *map space* M with a structural map $X \times M \to Y$. That structural map has a universal property that implies that the punctual elements of M are in bijective correspondence with maps $X \to Y$ in such a way that the structural map effects evaluation. I use the exponential notation $M = Y^X$.

To realize Euler's vision, I will postulate a space T of infinitesimals with certain properties (whose consistency has been shown many times) and then derive properties of the reals R defined as ratios of T.

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But what are ratios? Much confusion has been caused by the notational presumption that division is an operation on the same footing as addition and multiplication. The slogan "you can't divide by zero" by no means disposes of the issues that arise, especially for variable quantities. (In fact, one can even divide by zero, if it is zero that one wants to divide, getting infinitely many answers.) In algebra, if s is a quantity in a ring A, the result $A[s^{-1}]$ of dividing by s is another ring with a homomorphism

$$A \to A[s^{-1}]$$

typically neither injective nor surjective, because the new ring consists of functions defined on an open subspace

$$\mathrm{spec}(A[s^{-1}]) \hookrightarrow \mathrm{spec}(A)$$

of the domain of the original quantities. (For example if A is a ring of polynomials in one variable, then $A[s^{-1}]$ consists of rational functions whose denominator is a power of s, and the open subspace excludes the zeroes of the polynomial s). This and many other constructions should suggest that disaster can result from naive presumptions that symbols like difference quotients necessarily denote anything unique (Lawvere, 1996).

Because division really refers to the possible existence of inversion to multiplication, we define a (possible) *ratio* to be a general map $A \to B$ and we say, in particular, that "b is divisible by a" if and only if a ratio r exists with $b = ra$:

$$\begin{array}{ccc} & E & \\ {}^{a}\swarrow & & \searrow^{b} \\ A & \underset{r}{-\,-\,-\!\!\to} & B \end{array}$$

(Thus divisibility is categorically dual to "belonging to" (Lawvere and Rosebrugh, 2003)). For example, these notions apply in a monoid (when $A = B = E$) or in a linear category (where composition is traditionally called multiplication); our special example will be in a nonlinear category, except that the objects involved are so small that all the maps around them are affine-linear.

In the development of Synthetic Differential Geometry (Kock, 2006) over the past 40 years it has usually been assumed that a smooth category contains a line object which actually has the structure of a ring R. Then the subspace D of first-order infinitesimals ($h^2 = 0$) plays a key role. In particular, all the maps $D \to D$ are uniquely represented by elements of R, precisely fulfilling Euler's principle. I show below that reciprocally R can be constructed from a non-coordinatized version T of D, thus achieving a foundation for smooth geometry that is even "radically synthetic" in the sense that all algebraic structure is derived from constructions on the geometric spaces rather than assumed.

There is a direct non-quantitative basis for the necessity of our infinitesimal space T. The most fundamental functor characteristic of smooth and analytic categories is the tangent-bundle functor. Because this functor preserves cartesian products and more, it cries out to be construed as a *representable* functor. By Yoneda's Lemma, a representing object is unique.

The tangent-bundle functor has been (explicitly or implicitly) fundamental in mathematical physics because motion is ubiquitous in the material world. Motion means that a thing is in a certain place and yet elsewhere in the same instant. The generic mathematical model is still the one that involves nilpotent elements; for example, the canonical commutation relation of quantum mechanics is at bottom Leibniz's rule for the derivative. However, other models may be included in the general treatment we give below, in which one might think of T as a generic instant which has a certain point 0 but does not reduce to it.

Therefore we postulate a pointed space T and call the map space X^T the tangent bundle of any space X, with evaluation at the point inducing the bundle map $X^T \to X$. Even the point of T is not really a given structure in the intended examples, for in them $1 \to T$ is unique (but far from being an isomorphism). Thus every actual map $T \to T$ would automatically preserve 0; however, there would still be non-punctual elements $E \to T^T$ of the map space for $E \neq 1$, so we define by pullback

$$\begin{array}{ccc} R & \hookrightarrow & T^T \\ \downarrow & & \downarrow \mathrm{ev}_0 \\ 1 & \xrightarrow{\;0\;} & T \end{array}$$

the subspace R of *Euler reals*. The object T^T has an intrinsic multiplication arising from composition, and the subspace R is clearly closed under it, so we automatically get "multiplication of reals" as an operation $R \times R \to R$. As often happens, multiplication is more fundamental than addition.

Thus R has the intrinsic structure of a monoid with 0. Moreover, it has always been commutative. To justify that commutativity seems difficult, though intuitively it is related to the tinyness of T, in the sense that even for slightly larger infinitesimal spaces, the (pointed) endomorphism monoid is non-commutative. Note that if we define

$$D_n = \{h \in T^T | h^{n+1} = 0\}$$

then actually $D_n \subset R$, and $D = D_1$ is the object considered in previous Synthetic Differential Geometry work; for that reason a possible isomorphism $D \xrightarrow{\approx} T$ can be thought of as a unit of time (T itself has no intrinsic multiplication). A canonical map $T^T \to R$, retracting the inclusion, is needed to define derivatives and is obtained (together with the commutativity!) by the assumption that the composite of the inclusion followed by the universally commutative quotient monoid is an isomorphism.

Cantor extracted from the cohesive and active world of mathematics a subuniverse of discrete and inert sets (which served not only for cardinal measurements but also as a featureless background in which, dialectically, mathematics could be modeled by Dedekind, Hausdorff, Moore, Frechet, and their twentieth century successors). This contrast can be modeled by a subcategory with a reflector $X \to \pi_0(X)$ satisfying

$$\pi_0(X \times Y) \xrightarrow{\approx} \pi_0(X) \times \pi_0(Y)$$
$$\pi_0(1) = 1$$

(Lawvere, 2007). In particular, the S in the subcategory should satisfy

$$S \xrightarrow{\approx} S^T$$

because these S are spaces of non-Becoming, i.e. spaces in which no motion is possible, not even infinitesimal motion. Typically such a components functor π_0 exists; in particular, any algebraic structure that a space might carry is reflected as a similar structure on its "set" of components.

Proposition 1 $\pi_0(X^T) = \pi_0(X)$ *for all* X *iff* $\pi_0(R) = 1$.

Assume $\pi_0(R) = 1$ (i.e. R is connected). But that leaves many possibilities for $\pi_0(U)$ where $U \subset R$ is the subgroup of invertible elements. The above constructions would also provide a basis for complex-analytic geometry and analysis; in that case we would have $\pi_0(U) = 1$. However the intuition for the real case involves a line which is bi-directional, so that

$$\pi_0(U) = Z_2$$

a multiplicative group of two elements. In all cases, we can consider U_+ defined as the kernel of the natural homomorphism $U \to \pi_0(U)$ (i.e. the component of the identity) as the group of *positive* elements of R.

Further axioms, implying that R has an intrinsic addition, will be discussed below. But first, assuming that R has an addition, we show how to use only a given subgroup P (such as the U_+ defined by using π_0) to derive the structure of an *ordered* rig, that is, the rig R will have moreover a subrig M of non-negative quantities (in general a rig has commutative multiplication and addition satisfying the distributive law, but not necessarily negation). Define

$$A = \{a \in R | a + P \subseteq P\}$$
$$M = \{\lambda \in R | \lambda A \subseteq A\}$$

Proposition 2 *If* P *is a multiplicative subgroup of a rig, then* A *is an additive monoid and hence* M *is a subrig of* R. *If, moreover, 1 belongs to* A, *then* M *is contained in* A *and* P *is contained in* M.

Therefore the relation defined by

$$r \leqslant s \text{ iff } \exists m \in M[r + m = s]$$

has the expected properties of an ordering. The elements h for which

$$0 \leqslant h \,\&\, h \leqslant 0$$

constitute an ideal that contains all nilpotent quantities.

If the ambient category is a topos with generic subobject $1 \to \Omega$ and if we denote by Ω^M the space of order-preserving maps (parameterizing the upclosed parts of M) the natural Dedekind-Yoneda map

$$M \to (\Omega^M)^{\text{op}}$$

is given by the ordering on M itself. This map is neither injective nor surjective; it maps into the inf-completion $V \hookrightarrow (\Omega^M)^{op}$ that plays the role of the nonnegative semi-continuous Dedekind reals and serves as the natural recipient for metrics in the category. (Similarly, R itself can be order-completed, but by using M in a two-sided way.)

There are two ways to insure that R has unique addition: one involves integration and the other differentiation.

1. Integration (or distribution theory) concerns smooth linear functionals. Experience with Taylor series motivates the presumption that for suitable smooth spaces, homogeneity should imply linearity. Thus if we define

$$\Phi(X) = \text{Hom}_R(R^X, R) \hookrightarrow R^{(R^X)}$$

as the space of those functionals φ satisfying just the multiplicative condition

$$\varphi[\lambda f] = \lambda\varphi[f]$$

for λ in R, then the assumed extensivity property of general integration

$$\Phi(X + Y) = \Phi(X) \times \Phi(Y)$$

specializes to

$$\Phi(2) = \Phi(1)^2.$$

From that, $+$ emerges as the unique homogeneous map $R \times R \to R$ which becomes the identity when restricted to both 0-induced axes $R \to R \times R$.

2. The other route from multiplication to addition goes via trivial Lie algebras. The kernel

$$\text{Lie}(R) \hookrightarrow R^T \to R$$

has a binary operation induced from multiplication since the evaluation/projection is a homomorphism. However this operation is usually called addition; the space of endomorphisms of Lie(R) for that operation is a rig that contains (the right action of) R as a multiplicative sub-monoid, so that if we postulate that R exhausts the whole endomorphism space, then R inherits a canonical addition.

The object T is often assumed to be an ATOM; this can be read: "amazingly tiny objectified motion." The objectified motion reflects the intuition that elements of the tangent bundle are arrows obeying differential laws of motion, etc. The qualification "amazingly tiny" refers to the important property (not used in the above discussion) that T is *tiny* with respect to the whole category. This tinyness is partly expressed by (Grothendieck's notation) $f^!$, the surprising additional right adjoint that some topos morphisms f have. The morphism in question is the essential one having $f^*(X) = X^T$, but the additional operation $f^!(Y) = Y^{1/T}$ leads to the representability of differential forms, of laws of motion, and of still unexplored higher infinitesimal constructions. Tinyness is relative to the universe, as crudely measured by the amazing right adjoint:

Proposition 3 *Given any object T in any Grothendieck topos ϵ, then ϵ is a subtopos of a larger one ϵ' where T becomes tiny. But conversely, the smaller ϵ may not be closed under $(\ \ )^{1/T}$, so that T is typically not tiny with respect to the smaller container ϵ itself.*

Tinyness permits a certain compromise with Robinson's idea that infinitesimal constructions should preserve logic; we must restrict ourselves to geometric logic (also known as coherent, positive, or dynamic logic).

Proposition 4 *If T is tiny and if A is any object equipped with a structure described in geometric logic, then A^T enjoys the same structure with the same geometric properties, as is seen by composing with the morphism to the classifying topos for the structure theory in question.*

Most of the known models for the above discussion are "infinitesimally generated" in the sense that starting from T, all objects of the topos (such as the algebra of operators on Hilbert space) are obtained by the functorial operations of exponentiation, limits, and colimits. Such is the remarkable blossoming of Euler's principle.

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