Open Problems in Topos Theory *

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For Martin Hyland and Peter Johnstone in honor of their sixtieth birthdays.

Here are seven problems that I have not yet been able to solve. Clarification on them would further advance work on topos theory and its applications to thermomechanics, to algebraic geometry, and to logic.

1 Quotient Toposes

The number of "surjective" geometric morphisms with domain a given Grothendieck topos is obviously not small. For example, for every given monoid, there is the morphism to the topos formed by the actions of that monoid on objects of the given topos. But how many are those morphisms that, by analogy with the topological special case, are called "connected surjective"? These actual quotients are determined by left exact comonads that are moreover idempotent. Is there a Grothendieck topos for which the number of these quotients is not small?

At the other extreme, could they be parameterized internally, as subtoposes are?

2 Subquotients and Idempotents

The (coherently) idempotent left exact endofunctors parameterize the subquotients of a given topos, because the splitting of idempotents in the left exact 2-category Lex can be analyzed

*Invited address at the 88th Peripatetic Seminar on Sheaves and Logic, Cambridge, UK, April 4, 2009 into two steps, as was shown by Pare, Rosebrugh and Wood. In one step the inclusion is left-adjoint to the projection (corresponding to a quotient), and in the other step the inclusion is right-adjoint to the projection (corresponding to an inclusion). Since the system of left exact idempotents corresponds also to certain left exact subcategories in their own right, it also has a lattice structure. Thus, given any quotient (indeed any subquotient) there is a smallest subtopos containing "it" (i.e. its inverse image), and similarly given any subtopos there is a smallest quotient for which the composite is an inclusion. The first of these closure operators clearly exists because the class of subtoposes is parameterized by a single object; the question of the existence of the second one is related to question (1) above.

What are the algebraic properties of this lattice equipped with two closure operators, retracting respectively onto the two kinds of special subquotients?

3 Boundaries of classes of models

Because the system of subtoposes of any topos is contravariantly parameterized by the points of an internal Heyting algebra, it is itself a co-Heyting algebra and thus supports a boundary operator that is defined by contradiction and that satisfies the Leibniz rule. Because subtoposes represent classes of models of positive extensions of a given theory, the effect of the boundary operator (and of the double negation core operator) on classes of models needs to be determined both semantically and syntactically.

4 The jump operator on levels within a topos

The essential subtoposes of a given topos were shown with Max Kelly to form a complete lattice. Each such "level" receives a restriction or truncation functor from the top level and this functor unites two opposite subcategories that, as categories in their own right, are the same topos. For each level n, the essentiality gives an idempotent left adjoint comonad $\mathrm{sk}(n)$ and the sheaf inclusion gives the idempotent right adjoint monad $\mathrm{cosk}(n)$. The levels provide one systematic way of measuring the complexity of objects; an object can be said to belong to level n if it is its own n-skeleton. In the topos of simplicial sets, levels coincide with dimensions. Besides the obvious ordering of levels, there is a stricter ordering defined as follows: Level n is way below level m if every n-skeleton is an m-coskeleton, i.e. every object belonging to level n is also in the subtopos of m-sheaves. For the topos of simplicial sets Michael Zaks showed over 20 years ago that above every level there is a smallest level that is way above it; for the lowest dimensions this jump operator amounts to adding 1, but for higher dimensions n it corresponds to $2n - 1$.

There is another fundamental topos related to classical constructions and combinatorial topology, namely the Boolean algebra classifier that consists of presheaves on the category of finite non-empty sets. Here again, the levels correspond to natural numbers (together with minus infinity), so that there are sufficient chain conditions to ensure that the jump operator exists. What is, in combinatorial or number theoretic terms, the way below relation for this basic topos?

Update: The partial proof is published. Several key examples of this problem have been solved by the team of Carolyn Kennett, Emily Riehl, Michael Roy, Michael Zaks, Journal of Pure and Applied Algebra, Volume 215, Issue 5, May 2011, Pages 949-961)

5 Coverings that admit averaging and microlinearity

If K is a rig, then the category of finitely-presented K-rigs is a co-extensive co-site, so that the product-preserving functors from it to sets form a topos, inside of which we can define a further subtopos. Namely, consider that an inclusion $A \to B$ is a (co)covering if there exists an averaging process from B onto A, i.e. a retraction in the category of A-modules. For K = the field $\mathbb{Q}$, the restriction to extension fields of this covering notion applies to arbitrary inclusions, yielding the Boolean topos that (as suggested by Galois) is the natural base topos for algebraic geometry. More generally, if the natural numbers are invertible in K and if G is a finite group operating on B with A as fixed subalgebra, there is clearly such an averaging process. Similarly these coverings imply that internally any nilpotent is a sum of first-order nilpotents. The topos of sheaves thus has some of the good properties of micro-linear objects, but it has the advantage of being a topos. The needed stability property of these coverings follows from the fact that the coproduct for A-algebras is actually a functor defined for the weaker structure of A-modules.

In general, what is the relation between micro-linearity and these coverings? Are there other examples where a Grothendieck topology can be defined in this way by averaging with respect to structure that is weaker than that classified by the site? Can something similar be done with C^∞ functions and distributions of compact support? How does the above particular example fit into the family of Grothendieck topologies traditionally useful in algebraic geometry?

6 How strong is the adjointness of fractional exponents?

In a presheaf topos, the number of ATOMs is small. In a general topos, there is for any finite set of ATOMs a lower topos of relatively discrete objects over which it is defined and for which the defining adjointness for those ATOMs and their corresponding fractional exponents is enriched. Can the class of all ATOMs be suitably parameterized and can we define a lower topos over which all these adjunctions are enriched? (Recall that ATOMs are objects D for which $(\ )^D$ has a right adjoint $(\ )^{1/D}$; the term (distinct from Barr's above Boolean notion) was used in Bunge's thesis to help characterize presheaf categories, but such objects occur in more general contexts where they sometimes play the role of Amazingly Tiny Objectified Motion, the fractional exponent being a special case of the additional functor, characteristic of a local geometric morphism, that Grothendieck denoted with a superscript exclamation point.)

7 The algebra of time

Given a map $D \to A$ in a topos where D is an ATOM, the category of objects X, equipped with a prolongation operator $X^D \to X^A$ along this map, is a topos of laws of motion that receives an essential geometric morphism from the given topos, enriched over the lower topos of D-discrete objects. There is a further topos, consisting of actions of A^D, and a functor $(\ )^D$ from the prolongation topos to this further topos, corresponding to the classical representation of higher order ODE's in terms of first-order ODE's on phase space. This process has a left-adjoint F enabling the parameterization of any solutions of those equations over a time interval U as a motion, i.e. as a morphism of prolongations with domain $F(U)$.

Can the Algebra of Time $F(U)$ be calculated in familiar terms?