Foreword

The study that was initiated by Birkhoff in 1935 was named "general algebra" by Kurosh in his classic text; the subject is also called universal algebra, as in the text by Cohn. The purpose of general algebra is to make explicit common features of the practice of commutative algebra, group theory, linear algebra, Lie algebra, lattice theory, et cetera, in order to illuminate the path for that practice. Less than twenty years after the 1945 debut of the Eilenberg-Mac Lane method of categorical transformations, the obvious possibility of its application to general algebra began to be realized in 1963; that realization continues in the present book.

Excessive iteration of the passage

$$\mathcal{T}' = \text{ theory of } \mathcal{T}$$

would be sterile if pursued as idle speculation without attention to that fundamental motion of theory: concentrate the essence of practice, in order to guide practice. Such theory is necessary to clear the way for the advance of teaching and research. General algebra can and should be used in particular algebra (i.e. in algebraic geometry, functional analysis, homological algebra et cetera) much more than it has been. There are several important instruments for such application, including the partial structure theorem in Birkhoff's "Nullstellensatz", the "commutator" construction, and the general framework itself.

Birkhoff's theorem was inspired by theorems of Hilbert and Noether in algebraic geometry (as indeed was the more general model theory of Robinson and Tarski). His big improvement was not only in generality: beyond mere existence of generalized points, he showed they are sufficient to give a monomorphic embedding. Nevertheless, in commutative algebra his result is rarely mentioned (although it is closely related to Gorenstein algebras). The categorical formulation of Birkhoff's theorem ([65] and [91]), like the pre-categorical ones, involves sub-direct irreducibility and Zorn's lemma. Finitely-generated algebras, in particular, are partially dissected by the theorem into (often qualitatively simpler) finitely-generated pieces. For example, when verifying consequences of a system of polynomial equations over a field, it suffices to consider all possible finite-dimensional interpretations, where constructions of linear algebra such as trace are available.

Another accomplishment of general algebra is the so-called commutator theory (named for its realization in the particular category of groups); a categorical treatment of this theory can be found in [78] and [56]. In other categories this theory specializes to a construction important in algebraic geometry and number theory, namely the product of ideals [51]. In the geometrical classifying topos for the algebraic category of K-rigs, this construction yields an internal multiplicative semi-lattice of closed sub-varieties.

In the practice of group theory and ring theory the roles of presentations and of the algebras presented have long been distinguished, giving a syntactic approach to calculation in particular algebraic theories. Yet many works in general algebra (and model theory generally) continue anachronistically to confuse a presentation in terms of signatures with the theory itself that is presented, thus ignoring the application of general algebra to specific theories, like that of $\mathcal{C}^\infty$-rings, for which no presentation is feasible.

Apart from the specific accomplishments mentioned above, the most effective illumination of algebraic practice by general algebra, both classical and categorical, has come from the explicit nature of the framework itself. The closure properties of certain algebraic sub-categories, the functorality of semantics itself, the ubiquitous existence of functors adjoint to algebraic functors, the canonical method for extracting algebraic information from non-algebraic categories, have served (together with their many particular ramifications) as a partial guidance to mathematicians in dealing with the inevitably algebraic content of their subjects. The careful treatment of these basics, by Adámek, Rosický, and Vitale, will facilitate future mutual applications of algebra, general algebra, and category theory. They have achieved in this book a new resolution of several issues that should lead to further research.

What is General Algebra?

The bedrock ingredient for all its aspects is the use of finite cartesian products. Therefore, as a framework for the subject, it is appropriate to recognize the 2-category of categories that have finite categorical products and of functors preserving these products. Among such categories there are the linear ones whose products are simultaneously co-products; that is a crucial property of linear algebra in that maps between products are then uniquely represented as matrices of smaller maps between the factors (though of course there is no unique decomposition of objects into products, so it would be incorrect to say inversely that maps “are” matrices). General categories with products can be forced to become linear and this reflection 2-functor is an initial ingredient in linear representation theory. However, I want to emphasize instead a strong analogy between general algebra as a whole and any particular linear monoidal category, because that will reveal some of the features of the finite product framework that make possible the more profound results.

The 2-category of all categories with finite products has (up to equivalence) three characteristic features of a linear category such as the category of modules over a rig:

1. It is "additive" because if $A \times B$ is the product of two categories with finite products, it is also their coproduct, the evident injections from A, B having the universal property for maps into any third such category.

2. It is "symmetric closed"; indeed $\mathrm{Hom}(A, B)$ is the category of algebras in the background B according to the theory A. The unit I for this Hom is the opposite of the category of finite sets. The category J of finite sets itself satisfies $\mathrm{Hom}(J, J) = I$, and the category $Hom(J, B)$ is the category of Boolean algebras in B. As dualizer, the case B = small sets is most often considered in abstract algebra.

3. It is "tensored" because a functor of two variables that is product-preserving in each variable separately can be represented as a product-preserving functor on a suitable tensor-product category. Such functors occur in recent work of Zurab Janelidze [55]; specifically, there is a canonical evaluation $A \otimes \mathrm{Hom}(A, B) \to B$, where the domain is "a category whose maps involve both algebraic operations and their homomorphisms".

A feature not present in abstract linear algebra (though it has an analogue in the cohesive linear algebra of functional analysis) is Ross Street's bo-ff factorization of any map (an abbreviation of "bijective on objects followed by full and faithful"), see [89] and [90]. A single-sorted algebraic theory is a map $I \otimes A$ that is bijective on objects; such a map induces a single "underlying" functor $\mathrm{Hom}(A, B) \otimes B$ on the category of A-algebras in B. The factorization permits the definition of the full "algebraic structure" of any given map $u \colon X \to B$ as follows: the map $I \to \mathrm{Hom}(X, B)$ that represents u has its bo-ff factorization, with its bo part the algebraic theory $I \to A(u)$, the full X-natural structure (in its abstract general guise) of all values of u. The original u lifts across the canonical $\mathrm{Hom}(A(u), B) \to B$ by a unique $u^{\#}$. This is a natural first step in one program for "inverting" u, because if we ask whether an object of B is a value of u, we should perhaps consider the richer (than B) structure that any such object would naturally have; that is, we change the problem to one of inverting $u^{\#}$. Jon Beck called this program "descent" with respect to the "doctrine" of general algebra. (A second step is to consider whether $u^{\#}$ has an adjoint.)

Frequently, the dualizing background B is cartesian-closed, i.e., it has not only products but also finite co-products and exponentiation, where exponentiation is a map

$$B^{op} \otimes B \to B$$

in our 2-category. This permits the construction of the important family of function algebras $B^{op} \to \mathrm{Hom}(A, B)$ given any A-algebra (of "constants") in B.

On a higher level, the question whether a given C is a value of the 2-functor $U = \mathrm{Hom}(-, B)$ (for given B), leads to the discovery that such values belong to a much richer doctrine, involving as operations all limits that B has and all co-limits that exist in B and preserve finite products. As in linear algebra, where dualization in a module B typically leads to modules with a richer system of operators, conversely such a richer structure assumed on C is a first step toward 2-descent back along U.

The power of the doctrine of natural 2-operations on $\mathrm{Hom}(-, B)$ is enhanced by fixing B to be the category of small sets, where smallness specifically excludes measurable cardinals (although they may be present in the categorical universe at large).

A contribution of Birkhoff's original work had been the characterization of varieties, that is, of those full subcategories of a given algebraic category $\mathrm{Hom}(A, B)$ that are equationally defined by a surjective map $A \to A'$ of theories. Later, the algebraic categories themselves were characterized. Striking refinements of those characterization results, in particular, the clarification of a question left open in the 1968 treatment of categorical general algebra [64], are among the new accomplishments explained in the present book. As Grothendieck had shown in his very successful theory of abelian categories, the exactness properties found in abstract linear algebra continue to be useful for the concretely variable linear algebras arising as sheaves in complex analysis; should something similar be true for non-linear general algebras? More specifically, what are the natural 2-operations and exactness properties shared by all the set-valued categories concretely arising in general algebra? In particular, can that class of categories be characterized by further properties, such as sufficiency of projectives, in terms of these operations? It was clear that small limits and filtered colimits were part of the answer, as with the locally-finitely-presentable categories of Gabriel and Ulmer. But the further insistence of general algebra, on algebraic operations that are total, leads to a further functorial operation, needed to isolate equationally the correct projectives and also useful in dealing with non-Mal'cev categories: that further principle is the ubiquitous preservation of Linton's reflexive co-equalizers, which are explained in this book as a crucial case of Lair's sifted colimits.

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