Synthetic Differential Geometry. Second Edition. *By Anders Kock.* Cambridge University Press, Cambridge, UK, 2006. $55.00. xii+233 pp., softcover. ISBN 0-521-68738-1.

Forty years ago it became apparent that certain categorical methods, devised for the study of algebraic and analytic spaces, can also be applied to smooth (C-infinity) spaces. The ramifications of this observation for continuum physics are still being worked out, but major advances were recorded in the 1991 Springer book *Models for Smooth Infinitesimal Analysis* by Ieke Moerdijk and Gonzalo Reyes; undergraduate textbooks on differentiable analysis that explicitly make use of the resulting insights were prepared by René Lavendhomme (1996) and by John Bell (1998). These three books, as well as the research papers by Marta Bunge, Eduardo Dubuc, Felipe Gago, and others, descend from the first book to treat these developments, *Synthetic Differential Geometry* (Cambridge, 1981) by Anders Kock; the volume under review is a revised and augmented version of that earlier book.

Several mathematicians have proposed improvements of the category of finite-dimensional smooth manifolds. Notably, K.T. Chen and Alfred Frölicher both contributed interesting proposals to volume 1174 of the Springer Lecture Notes in Mathematics, which records a 1982 meeting at Buffalo. Frölicher's admirably simple category is the basis for the book *The Convenient Setting of Global Analysis* (AMS, 1997) by Andreas Kriegl and Peter Michor. However, that category does not permit the "representability" of tangent vectors and differential forms. Chen's category, on the other hand, was explicitly designed to allow representability of differential forms, but does not include representability of tangent vectors, nor the representability of smooth function spaces. Remarkably, all these representabilities and more do follow from the "topos" formulation expounded in the book under review. The practical significance of representability is that many constructions of a naively set-theoretic character are not mere formulas, but correspond to actual geometric objects, indeed objects with an automatic smooth structure.

Not only diffeomorphisms, but paths, functions, subspace inclusions, product projections, etc., are used in differential theories, and composition of these leads explicitly to the idea of a category; but various relevant categories arise, depending on which objects are admitted and which functorial constructions on them are possible. The variety of categories also arises from the fact that useful levels of smoothness (polynomial, analytic, or an intermediate level such as the one generated by bump functions) all give rise to categories whose general properties and geometric uses are somewhat similar, so that a common nontrivial theory is possible. We refer here to the categories described by that theory as "smooth toposes."

Besides the usual cartesian products (used by Galileo for the description of motion), two of the striking features of smooth toposes are the representability (described more fully two paragraphs below) of tangent bundles and of function spaces. For 300 years mathematical scientists such as Bernoulli, Euler, Lie, and Cartan have used these functors and even their representability, despite the fact that such representability was inconsistent with the formal foundations that were available. The issue in all the classical cases is to broaden the definition of the spaces sufficiently to permit the desired representabilities, thereby streamlining constructions and computations without obscuring the spaces and mappings of interest.

By Yoneda's lemma, representing objects are unique, in particular the object T, which represents the tangent bundle of any space X as the function space X^T; T is first-order infinitesimal in the sense that the kernel of the bundle projection $R^T \to R$ for the ring R of reals consists of quantities of square 0. Elementary considerations show that the natural properties of T are "inconsistent with classical logic," not in any mystical intuitive way, but just in the sense that the lattice of subspaces is not Boolean. It should be clear from studying this book that it was the development of the geometry that led to "logical" considerations such as non-Boolean subspace lattices, double-negation of points, explicit distinctions between theories and models, etc.; researchers in this field did not set out to impose a preconceived version of logic onto the subject. In particular, the nilpotent infinitesimals as representors for the geometric tangents should not be confused with the infinitesimals invoked in "nonstandard" systems of logical formulas. Euler went so far as to affirm that real numbers themselves are ratios of infinitesimals; the fact that R is the zero-preserving subspace of T^T now gives rigorous justification to his idea.

The representability of function spaces in a smooth topos implies a unique notion of smoothness for functionals. An example is the smooth space of all V-valued distributions of compact support on X, where V is a space with an action of the multiplicative monoid R. This is just the subspace of the double function space V to the power (R to the power X) consisting of the R-homogeneous smooth functionals. (As Kock and Reyes showed in their recent work on the heat equation, noncompact distributions can also be handled.) But what precisely is meant by representability of function spaces? This special case of Daniel Kan's concept of an adjoint functor means that given any three spaces B, I, S, there are other spaces S^I and S^B for which maps $I \times B \to S$ correspond uniquely to maps $B \to S^I$ and also to maps $I \to S^B$; if B, I, S denote a body, a time interval, and a space region, respectively, then we have the well-known bijection between placements in path space and paths in placement space. These function spaces permit the calculation of the velocity field on B and the motion of the center of mass of B, by composition with basic functionals of differentiation and integration.

Around 1980 it was realized that much stronger representabilities follow automatically in all the constructed examples of relevant categories. It had already been clear in 1967 that the category of first-order ODE in a smooth topos would again be a smooth topos (with all the representability and exactness that provides); here a first-order ODE (or vector field) on X is equivalent to a pointed action $T \times X \to X$. Indeed the pointed actions of any given pointed space, with the natural morphisms of such actions, form a category as good as the ambient category of spaces. But as expressed by the property that T is an ATOM ("amazingly tiny objectified motion"), the stronger representability implies the nonclassical result that the category of nth order ODE is also a topos. Here n is a given map $T \to A$ (where, for example, $A = T^2/2!$ is the representor for the symmetric second tangent bundle functor); an nth order ODE on X is a given section of the induced $X^A \to X^T$ (in the example, a given way of prolonging any linear infinitesimal path to a quadratic one); and solutions are morphisms in the resulting category having time-like domains. In their work on the wave equation and the heat equation, Kock and Reyes have explicitly utilized this notion of second-order equation as applied on infinite-dimensional spaces of distributions.

In the above brief sketch we have omitted many aspects, such as affine connections, the Thom–Mather singularity classification, the role of nonnilpotent germs, and the Stokes and DeRham theorems for currents, all of which have been revealed in a fresh light by these synthetic considerations. The "synthetic" method, developed from the geometric tradition of Sophus Lie, (1) starts with an objective concept and (2) describes it axiomatically, then (3) constructs models for those axioms by using numerical coordinates or even logical power sets, and finally (4) uses both the axioms and the models to further illuminate and make precise the concept. An informed modification of the axioms, or a refinement of the concept, may prove necessary. For the objective concept of spaces as arenas for smooth motion, one discerns the role of all these synthetic stages in Anders Kock's book.

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Partial Differential Equations of Applied Mathematics. Third Edition. *By Erich Zauderer.* Wiley-Interscience, Hoboken, NJ, 2006. $115.00. xxviii+930 pp., hardcover. ISBN 0-471-69073-2.

Among the many partial differential equation (PDE) textbooks for advanced undergraduates and graduate students studying applied mathematics or engineering, Erich