Strong Nullstellensatz for Euler continua in cohesive toposes

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The axiomatic theory of cohesive toposes is based on common features of algebraic, analytic, and smooth geometry/topology and of combinatorial toposes like simplicial sets (1950) and reflexive graphs. The aim is to make explicit a background for dynamical models of the motion of bodies and waves. Such toposes E are defined over other toposes K considered to be (relatively) non-cohesive, for example K might be "the" Galois or "the" Cantor topos; there is given a product-preserving functor from E to K considered to "count cohesive pieces" which determines a string of right adjoints pieces, discrete, points, codiscrete. The assumption "the natural map from discrete to codiscrete in E is monic" is a (weak) Nullstellensatz because it is equivalent to the epimorphicity of the corresponding map from points to pieces in K. In commutative algebra a stronger Nullstellensatz (in the spirit of Birkhoff) affirms that the homomorphisms from a given function algebra to "infinitesimal" algebras are jointly monic; a dual idea is that for certain spaces R in a topos (but not all), an equation between two maps from X to R would follow from their equality on each infinitesimal figure in X. Of course nilpotents and ATOMs play key roles, but in the present context a natural broad definition of "infinitesimal" captures those spaces for which every piece has a unique point. (For example, all arrows in an infinitesimal reflexive graph are loops). Every space then has a maximal infinitesimal subspace, which is its substance or canonical intensive quality (in the sense of my CT06 contribution, "Axiomatic Cohesion" to appear in TAC). A space R satisfies the strong Nullstellensatz relative to this notion of infinitesimal if substantially equal R-valued functions are equal. The spaces satisfying this strong Nullstellensatz are closed under exponentiation by infinitesimals: given that $R = B^A$ where A is infinitesimal, if B satisfies the strong Nullstellensatz, then so does R. Euler described reals as ratios of infinitesimals; because spaces of ratios are retracts of map spaces, it follows that the strong Nullstellensatz holds for Euler continua in E if it holds for the infinitesimal spaces themselves.