ARTICLE IN PRESS
S0315-0860(04)00059-X/BRV AID:2460 Vol.●●●(●●●)
YHMAT:m2 v 1.24 Prn:24/09/2004; 10:35
yhmat2460
[DTD5] P.1 (1-7)
by:Vita p. 1
Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Historia Mathematica ●●● (●●●●) ●●●–●●●
HISTORIA MATHEMATICA
www.elsevier.com/locate/hm

Review

Hermann Günter Grassmann, A new branch of mathematics, "The Ausdehnungslehre of 1844," and other works, Translated by Lloyd C. Kannenberg, with foreword by Albert C. Lewis, Open Court, 1995.

Hermann Günter Grassmann, Extension Theory, "The Ausdehnungslehre of 1862," Translated and with a foreword and notes by Lloyd C. Kannenberg, History of Mathematics, vol. 19, American Mathematical Society/London Mathematical Society, 2000.

In the 1990s there was a resurgence of interest in the mathematical work of Hermann Grassmann (1809–1877), who was a versatile and productive scientist and a Stettin high-school teacher all his professional life. This new interest gave rise in particular to the first English translation of his two books, called briefly A_1 and A_2, which are being reviewed here. Other closely related recent publications are: the French translation [Grassmann, 1994] of A_1 by Dominique Flament; also by Kannenberg, the translation [Peano, 2000] of Giuseppe Peano's 1888 "Calcolo Geometrico," which Peano had advertised as a presentation and simplification of Grassmann's theory; and the Proceedings of the 1994 Insel Rügen sesquicentennial conference of scholars from diverse specialties, organized by Gert Schubring.

The translations

An important virtue of Kannenberg's translation of A_1 is that he has included in the book the translations of several selected articles in which Grassmann had used the "Ausdehnungslehre" to analyze Hamilton's quaternions and to formulate electrodynamics and mechanics. Particularly helpful is Grassmann's 1845 "Brief Survey of the Essentials of Extension Theory" (pp. 283–295). Also included is a translation of a 1679 letter from Leibniz to Huygens which contains an early formulation of a program that enjoyed a qualitative advance, all in the brief period from 1823 to 1844, at the hands of Bellavitis, Hamilton, Grassmann, and Möbius. Briefly, this program is to develop an algebra which directly refers to and combines geometric figures and their motions, with arbitrary Cartesian coordinates relegated to their properly auxiliary role.

The modern mathematical reader will unfortunately find some of the terminology of both reviewed books unusual because the translator has chosen, for example, to uniformly translate "Verknüpfung" as "conjunction," even though modern usage would usually call for "operation," or occasionally for "connection" or "combination." The French translator mentioned above has provided an extensive two-way glossary of the French–German correspondence that he chose. Thus, guided by both the French and the English versions, a renewed and intensive study of the original German has become possible.

doi:10.1016/j.hm.2004.07.004

ARTICLE IN PRESS
S0315-0860(04)00059-X/BRV AID:2460 Vol.●●●(●●●) yhmat2460 [DTD5] P.2(1-7)
YHMAT:m2 v 1.24 Prn:24/09/2004; 10:35 by:Vita p. 2

Is such a study worthwhile at this late date? The featured AMS Review [MR 2001d: 01048] of this AMS publication of A_2 gives the impression that such a study is of merely historical interest. On the other hand, my colleague Stephen Schanuel and I had previously found, during an initial study of the *Ausdehnungslehre*, that there are several mathematical results in it which should be known to present-day mathematics, but are not.

Difficulties, real and alleged

A serious stumbling block to the study which is needed for extracting these mathematical results and developing them further has been Grassmann's German writing style. Grassmann had criticized Hegel for an arbitrary unclarity in his philosophical discussion of mathematical issues; that criticism has struck many as a case of the pot calling the kettle black, since even German-speaking mathematics students have found the language of A_1 difficult. The gargantuan efforts of the translators should become an important aid to those students as well.

Another stumbling block has been a mathematical misconception, which I will describe below, emanating from the 1894 editors' footnotes to A_1 (p. 300) (originally published with the *Collected Works* of Grassmann).

From the beginning it has been widely claimed that the main stumbling block is Grassmann's philosophical introduction (A_1, pp. 23–43). The last half of that introduction is essentially one of the first expositions of the rudimentary principles of what today might be called universal algebra. The content of the first half, after considerable study of the compact formulations, appears to be a simple and clear natural scientist's version of the basic principles of dialectical materialism, as applied to the formal sciences. Nonetheless, the reputation of Grassmann's work as mystical and mysterious became widespread.

Sometimes the popularization of Grassmann was not motivated by love of geometry, nor aimed toward clarification of learning, development, and use of that science. The presumption of the difficult character of Grassmann's work was used for other purposes.

In Chicago, Paul Carus, the founder of the Open Court Publishing Company, edited *The Monist* from 1890 to 1919; the journalistic policy was to exploit recent scientific results (not yet widely understood by the public) to cast doubt on science and thus to rescue religious speculations from the advance of science.[1] In that milieu Grassmann's work became subjected to the same abuse that was shortly to befall relativity and quantum mechanics.

In Turin, Italy the application of the Ausdehnungslehre to geometry was already well under way in 1883 by Corrado Segré and his school, which eventually included Veronese, Castelnuovo, Enriques, and others as described by Aldo Brigaglia in [Rügen, pp. 155–164]; nonetheless, in 1888 Giuseppe Peano suggested that the supposed incomprehensibility of Grassmann's geometric calculus could be alleviated

[1] This method, borrowed not unwittingly from Bishop Berkeley, led to a tortured definition of "science" that permitted Carus to exult after the World Parliament of Religions (Chicago, 1893) that Buddhism is the "most scientific" of religions. His name is well known to mathematicians as the title of a series of expository monographs (Carus Monographs) issued by the Mathematical Association of America; that series has been self-supporting for most of its life due to the mathematical and pedagogical virtues of its contents. Not so well known is the service, going well beyond the mere perpetuation of the name, in return for which Carus' widow provided the original seed money: at the same 1922 meeting of the Association where the grant was announced, the retiring address of the president had as its sole theme the claim that the acceptance of the mathematical concepts of infinity, infinitesimal, and the fourth dimension necessitate also the acceptance of the ideas of God, individual insignificance, and heaven.

by adopting Peano's own version of logic. Relying on strings of symbols while rejecting the objectivity of geometric figures became the remedy recommended even in high-school texts.

A more mathematically oriented development of the algebra of logic was advanced, for example, by Ernst Schröder, who received significant direct inspiration from the writings of Grassmann and his brother, as Volker Peckhaus has detailed in [Rügen, pp. 217–231].

In spite of the interferences, the continued persistence of physicists and mathematicians has been responsible for the fact that a large portion of the mathematical content of the *Ausdehnungslehre* has been understood and further developed and is being applied daily. Now the advances of modern mathematics and these translators have put us in a position to understand even more.

The philosophy

Many eminent scientists in history scrupulously separated their scientific work from their religious views, if they had any; Grassmann seems not to have deviated from that policy. However, the attempts by others to link his work to theology has suggested to some scholars (such as Engel, an editor of the collected works) that the 10-page philosophical introduction to A_1 is just a reflection of the *Dialektik* of Friedrich Schleiermacher, a pillar of the Prussian church whose lectures at the University of Berlin Grassmann had followed. However, according to the research of Gert Schubring [Rügen, 1996, pp. 59–70], Schleiermacher cannot have been the sole decisive influence. Engel's claim of such a decisive influence had been based on Grassmann's own youthful response to the theological examiners for the ministry; according to Schubring's investigation, Grassmann never became a minister, and his response to the examiners for science teachers' credentials was quite different. His brother Robert and his father Justus, active mathematicians and philosophers in their own right, were much more pervasive influences.

Concerning the introduction to A_1, Grassmann insists that his reason for including it is an attempt to provide an orientation to help the student form for himself the proper estimation of the relation between general and particular at every stage of the learning process. His formulation that philosophy moves from general to particular, and mathematics from particular to general, can be traced to Kant and probably was present in some form in Schleiermacher's lectures. But especially his original use of the pair of fluid oppositions, "continuous" versus "discrete," and "equality" versus "difference," with their dialectical development into a basic fourfold structure within the mathematical sciences, has been quite suggestive to mathematicians who have studied it. Had the editors of his collected works taken seriously his distinction between Becoming and Change, they might not have fallen into the misconception that his simple laws of becoming were mere vectors. (In the translation of the philosophical introduction "Sein" and "Werden" have been rendered as "the existent" and "continuous evolution," rather than as the more standard "Being" and "Becoming.")

The most basic sense of "dialectics" as Grassmann applies it seems to be this: in order to understand a situation which unites two opposing aspects, the first program is to recognize each aspect and the relation between them, rather than to set out from the beginning to prove that one aspect is everything and the other one is nothing. A very relevant case of that principle was enunciated in the 1832 statement by the Swiss geometer Jakob Steiner (who was Grassmann's predecessor in his first teaching job): "Neither the synthetic nor the analytic method constitutes the essence of the matter, which is the discovery of the dependency of forms on each other and the manner in which their properties are continued from the simpler figures to the more complex" (reprinted in E. Cassirer, *Substance and Function*, Dover, New York, 1953, p. 78).

Some of the geometry

Grassmann is commonly recognized as the discoverer of linear algebra. Thus Grassmann knew about vector spaces V (though he spoke of laws rather than of axioms), but he also knew about the affine linear spaces E which have no distinguished origin and in which only those linear combinations (of points) can be formed whose coefficients add up to 1. The geometrical linear algebra is not about either of these alone, but rather about both and their relationship. That relationship has as one aspect the action

$$E \times V \longrightarrow E$$

(denoted by $+$) whereby points are translated by vectors into other points; the other aspect, which satisfies the "torsor" property, is subtraction,

$$E \times E \longrightarrow V.$$

A vector space can be considered as an affine space equipped with the additional structure of a given point called the origin, and an affine space can be considered as a vector space (of one dimension higher) equipped with the additional structure of a given linear "weight" functional w, with the identification $E = w^{-1}(1)$. In the latter picture the action of the translation vectors $V = \ker(w)$ becomes a special case of the addition in the larger vector space $G_1(E)$.

Grassmann explains beautifully some of the elementary physics of flotation by systematically using the following facts, which at first seem to be paradoxical: A distribution on $G_1(E)$ integrates to an element of $G_1(E)$ whose w-value may be called its total weight. If that total weight is an invertible scalar, then the distribution can be normalized to give an actual point called its center, but if the total is zero, no normalization is possible and indeed the element itself is a vector, a pure becoming, which seems very unlike the pure beings that are points. A floating body has two distributions: a mass distribution and a volume distribution, each having nonzero total and hence its own center point. These give rise to gravity and buoyancy fields whose difference has zero total, and hence gives rise to a vector. This emerging vector connects the two center points and reveals the orientation that the floating body assumes.

The above E versus V story is nowadays fairly well known, but there may be a tendency to regard it as a mere detail of only pedagogical interest. It is in fact crucial to understanding A_1, because the "continuation of the simpler figures to the more complex" involves functors which may transform apparently minor differences into more profound ones having considerable conceptual and computational content. Thus the nowadays fairly well-known statement that Grassmann algebra is exterior algebra is so oversimplified as to be misleading.

The exterior algebra $\Lambda(V)$ of a 3-dimensional vector space V is 8-dimensional, but the Grassmann algebra of a 3-dimensional space is 16-dimensional (A_1, p. 289) and, moreover, has a highly nontrivial ingredient ∂ of structure that the exterior algebra does not have; this all stems from the fact that the Grassmann algebra functor G applies to affine linear spaces E and is functorial with respect to affine–linear maps. Again, there are strong relations: one can consider that

$$G(E) = \Lambda(G_1 E),$$

S0315-0860(04)00059-X/BRV AID:2460 Vol.●●●(●●●) yhmat2460 [DTD5] P.5 (1-7)
YHMAT:m2 v 1.24 Prn:24/09/2004; 10:35 by:Vita p. 5

where G_1 forms the vector space freely generated by (and hence having dimension one greater than) the affine E. Since $G_1(1) = R$ (the space of constant scalars), the unique affine–linear map $E \longrightarrow 1$ to the one-point space induces a linear weight functional $G_1(E) \longrightarrow R$, and the Koszul complex construction (known to Grassmann, though of course not by that name) extends (A_1, p. 179) that linear functional to the whole graded algebra $G(E)$ as the operator ∂, which Grassmann calls the "Ausweichung." This boundary operator satisfies the (signed) Leibniz product law and itself has square zero, while the elements P of $G(E)$ which have $\partial P = 1$ are just the points of the original space E. The kernel of ∂ is the exterior algebra of the translation vector space V of E. These "extensive quantities" in $\Lambda(V)$ thus act on the "rigid" figures in $G(E)$ (triangles, tetrahedra, etc.) in a way that extends the action of V on E. If we identify any given point P with the operation (of degree 1) on $G(E)$ of forming the Grassmann product $P()$ with it, we obtain a splitting of ∂, i.e., $\partial P \partial = \partial$ and $P \partial P = P$. Grassmann showed that the ∂-sequence is exact (A_1, p. 181); indeed if $\partial x = 0$, then $x = \partial(Px)$ for any point P.

The misconception of the 1894 editor (A_1, p. 300), to the effect that Grassmann's simple laws of becoming can only be mere translations, has obscured the fact that the condition of anti-symmetry on the Grassmann product, appearing in most modern treatments as a convenient imposition, actually has an independent conceptual basis within the affine category itself. The simple laws, interpreted as internal actions in the monoidal category of affine linear spaces, are just those flows generated by an affine–linear map S, yet affine–linear also in the time variable, which turns out to imply that S^2 equals the affine combination $2S - I$. Any translation is indeed such an S, but more generally, shear transformations (those whose derivatives differ from the identity by a transformation of square 0) are simple. Since for any three noncollinear points there are simple laws S which move one point to another while fixing the third, his definition of equality of axial vectors in terms of simple laws is equivalent to the anti-symmetry requirement [Rügen, pp. 262–264].

Grassmann systematically surveyed the possible congruence relations that can be naturally imposed on the free associative algebra generated by a vector space. Without naturality conditions every possible algebra would so arise. While group theory is usually invoked, actually in many cases the stronger functorality with respect to noninvertible maps is needed. Also, Grassmann emphasized that in studying a given space, other spaces of other dimensions inevitably arise, so that the domain of such an algebraic construction is for practical purposes a category of spaces and maps. But several such categories need to be considered; the more structure carried by the objects, the fewer the maps that preserve it, and hence the less restrictive is the naturality requirement on constructions. Thus if the domain category consists of linear spaces equipped with an anti-associative binary operation (Lie algebra), then a natural quotient is the enveloping algebra construction, whereas on the category of linear spaces equipped with a quadratic form, a natural quotient is the Clifford algebra construction, which includes Hamilton's quaternions as a key example. The above-sketched role of the simple laws of motion shows that even the structure of a given linear form permits a natural quotient G to be defined. This relationship should be given a pedagogically effective presentation for students, because although the anti-symmetric multiplications can be shown to express well various particular relations, the conceptual origin of the anti-symmetry itself can remain mysterious. Not only electromagnetism, but also several aspects of mechanics become more unified if certain key quantities are considered as axial vectors, rather than as mere vectors. An example emphasized by Schanuel is that the law of conservation of angular momentum is actually independent of any concept of angle, but is rather a special case of the conservation of a single momentum considered as an axial vector. It is important that a functor may have more structure than its codomain specifies; for example, the ∂ operator is a natural structure on G but Λ has no such natural structure.

ARTICLE IN PRESS
S0315-0860(04)00059-X/BRV AID:2460 Vol.●●●(●●●) yhmat2460 [DTD5] P.6 (1-7)
YHMAT:m2 v 1.24 Prn:24/09/2004; 10:35 by:Vita p. 6

The modern conceptual apparatus, involving levels of structure, categories of morphisms preserving given structure, forgetful reduct functors between categories, the adjoints to such functors, etc., seems to be necessary for ordinary mortals to be able to find their way through the riches of Grassmann's geometry. For example, inspired by the late Gian-Carlo Rota, Andrea Brini, and Antonio Teolis [Rügen, pp. 231–242] have described how the additional structure of a given volume form, or isomorphism of the constants R with the top-grade part $G_{n+1}(E) = \Lambda_n(V)$ of the Grassmann algebra, induces an alternating bilinear form on the whole algebra $G(E)$ and hence a second product that is a linearized intersection in the sense that the Grassmann product is a linearized union. The idea of linearized union is related to the Grassmann manifolds that arise on projectivizing the linear category (i.e., on taking the quotient category modulo central units).

The volume-form structure is carefully distinguished from the specification of a symmetric bilinear form on V itself (and hence on each $\Lambda_k(V)$). Such a metric structure is treated by Grassmann in A_2 by the novel method of considering first the weaker structure of perpendicularity, by postulating to every linear subspace a distinguished supplement or "Ergänzung." Since kernels exist, this algebraic expression of the intuition of "shadows" could equivalently be achieved by a structure on the linear category which assigns a distinguished splitting g to every map f (i.e., $fgf = f$ and $gfg = g$). A full metric structure is equivalent to a contravariant involutory functor $(\ )^*$ on the linear category, and in real cases there is, for every map f, a factorization $f = ip$ such that both i^*i and pp^* are invertible; composing these two self-adjoint automorphisms gives an automorphism θ of the middle ("rank") object such that $g = (i\theta p)^*$ is a splitting of f as required for a perpendicularity structure.

An important achievement of Grassmann was the demonstration that the construction of cubic and higher-degree curves and surfaces can be efficiently carried out using his "lineale" method without any use of coordinates. Clearly, that construction is essentially the formation of fibers (or equalizers) of multilinear maps that have been diagonalized (i.e., have some variables set equal). But the monoidal product on the category of affine spaces (like its close relative, the tensor product of vector spaces) does not have a diagonal map. The required nonlinear extension of the notion of map can be achieved by passing to the minimal category in which the given product functor "becomes" the categorical (Cartesian, or Galilean) product; the result is the category whose objects are commutative coalgebras!

I hope that in this brief review I have been able to give a glimpse of the material in the Ausdehnungslehre that still needs to be clarified for, and applied by, modern mathematics and physics. Lloyd Kannenberg's admirable achievement, qualitatively improving the accessibility to the English-speaking world, will assist in making that needed leap forward possible.

References

Grassmann, H.G., 1994. La science de la grandeur extensive, traduction et préface de Dominique Flament et Bernd Bekemeier. Edition Albert Blanchard, Paris.

Peano, G., 2000. Geometric Calculus. Translated by Lloyd C. Kannenberg. Birkhäuser, Basel.

[Rügen] Schubring, G. (Ed.), 1996. Hermann Günther Grassmann (1809–1877), Visionary Mathematician, Scientist and Neohumanist Scholar, Papers from a Sesquicentennial Conference, Boston Studies in the Philosophy of Science, vol. 187. Kluwer Academic, Dordrecht.

F. William Lawvere
SUNY at Buffalo,
Buffalo, NY 14260-2900,
USA