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Left and right adjoint operations on spaces and data types

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For Dana Scott's Seventieth Birthday

Abstract

Formal theories of higher types need to be augmented to permit treatment of some additional right adjoints which arise in mathematical practice. The concomitant mathematical scrutiny of supposedly well-established left adjoints resurrects a "critique of foundations" which may not be irrelevant for practice. Categorical methods based on mathematical experience with the diversity of toposes allow us to address such problems without getting entangled in sterile ontological debates.

An important categorical contrast is the reflection of objective qualities into quantitative algebra by means of the Steiner–Cantor–Burnside–Grothendieck–Schanuel abstraction process which replaces a category with its set of isomorphism types and simultaneously replaces adjoint functors with appropriate operations such as addition and multiplication. When products distribute over coproducts in a category of spaces, a rig structure is obtained which receives useful invariants of the spaces, as shown by Schanuel [13] and others. (The short-term "rig" is intended to replace the unwieldy "commutative semiring with 1"; note that one can adjoin to rigs a negative operation (= "n") to obtain rings, using the left adjoint to the obvious inclusion functor.)

Often adjoint functors beyond addition and multiplication can be taken into account as operations. Among the many reasons for seeking invariants that help to survey the complicated diversity of categories of spaces is the proposal that such categories might also serve as theories of data types. In order to bring that proposal closer to serving as a guide to actual data processing, such additional operations might play a role.

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1. "Constructive" left adjoints

Operations of the kind 1/1-A are inexplicably described as merely formal in some computer science literature. But that overlooks the availability of perfectly rigorous treatment of distributive categories, of rigs, and of the relation between them. Solutions X of "equations" like

$$X = 1 + A \times X$$

can be studied in terms of given isomorphisms in appropriate distributive categories. Such isomorphisms functorially induce further isomorphisms

$$X = 1 + A + A^2 + \cdots + A^n + E_n$$

for any given n. Thus any solution of the equation is an idealized model of "lists in the alphabet A", to the extent that if we take n large enough and make no special assumption about the infinite error term $E_n = A^{n+1} \times X$, we might be able to draw some conclusions about actual finite lists.

However, the 20th century tradition was to go far beyond this mild idealization by making the very strong assumption that the solution X is a smallest one, in the sense of the universal mapping property of the left adjoint on the given category to the forgetful functor from A-automata in the *same* category. Those who are overawed by the mythology of constructivism might believe that this way of determining a preferred solution of the equation is the "simplest". (A special case is the uncritical acceptance of the term "natural" for an alleged object that contains "exactly" $0, 1, 2, 3 \ldots$.) As is implicitly well-known, minimality is not so much a question of "infinity" or "size" but rather of the availability of sufficiently complicated means $X \to Y$ to show that X is indeed "smaller" than any $1 + A \times Y \to Y$. In other words, this particular sort of left adjointness generates one very complicated category which promises all the recursion anyone will ever want to do. But, in practice, one wants to do only a certain class of recursions; why not accept many thinner categories and their relations as a more accurate model than a single super complicated one?

The computers of any generation, no matter how advanced, will (beyond some n) have no direct information about the lists in the error term E_n, nor any way of dealing with them. This sort of consideration suggests that determining X as "generic" rather than as minimal might be more accurate; or in other words we might minimize properties rather than "size". The convenience of objectifying subjective infinity can be had in a more controlled fashion. Schanuel's polyhedral spaces [13] demonstrated that in the case $A = 2$, great simplification can result. I conjectured that the same would be true for the nonlinear tree equation $X = 1 + X^2$ just on the basis that I could functorially deduce from it that (like the sixth root of unity) $X^7 = X$, but could not show, for example, that $X^3 = X$; then Blass [1] was able to show that in the generic case there really are no perfect parameterizations of tuples of trees by trees except for the rig-theoretically "obvious" exponents like 7. The precise meaning of "generic" is actually also a left adjoint, but in a certain 2-category, where the answer is a much thinner category: one considers all distributive categories equipped with a chosen object and a chosen isomorphism making the object a solution of the equation in question, these categories being compared by appropriate functors. Gates [4] generalized and systematized the work of Schanuel and Blass, by considering fixed points for polynomial functors of all degrees.

As I pointed out in 1964 [8], the usual properties of recursion follow just from the universality associated with a left adjoint. In general, pure algebraists fruitfully imagine that such free algebras really exist, but the actual computer manipulation of the symbols involved in the presentation of algebras may be better modeled by categorical relations that are different from left adjointness. As workers in the O-minimal theory of real analysis [10] have noted, the so-called "natural" number object is the basis of many constructions and statements (such as Peano's space-filling curves and Gödel's incompleteness theorem) whose interference in the pursuit of geometry and analysis can actually be partly avoided.

2. Exponentiation in categories and rigs

Let us instead consider right adjoint operations. The most important are the exponentials which in particular objectify higher types. Implicit for centuries, they became more explicit with Volterra's theory of functionals and still more explicit with Hurewicz's definition of the k-spaces that he needed for homotopy theory. Definitions given by Hurewicz in his 1948–1949 lectures at Princeton were used by Gale [3]. Apparently, independently of those mathematical developments, combinatory logics were developed. Kan pointed out that exponentials are characterized as right adjoints; this fact was used axiomatically in the 1963 treatment of the category of categories [7]. When Eilenberg and Kelly systematized the theory of general closed categories in 1965, they noted the important special cases in which the tensor product is the intrinsic categorical product, traditionally called the "cartesian" product. Thereafter, categories with exponentiation have often been called "cartesian closed categories".

Categories with (sums and products and) the property that exponentials exist give rise, by the abstraction process, to structures which could be called exponential rigs. These satisfy what Tarski called the "High School Identities" and also carry a partial ordering induced from the existence of retractions in the category. (Tarski excluded 0, but here we include it and assume $0^0 = 1$.) Tarski asked whether the category of finite sets is generic for that theory. In recent years his question has been answered negatively. However, the counterexamples do not seem to shed much light on the questions: "Which exponential rigs arise objectively?" and, in particular, "Which arise from Dedekind-finite toposes?" There are many such toposes of Dedekind-finite objects that are quite concrete, namely the "finite toposes" of finite-set-valued presheaves on (=actions of) a finite category; the answer is not obvious because the exponentiation (in spite of being intrinsic), is not preserved by the natural comparison functors between examples (except in the case of group actions).

In an infinite realm, Dana Scott showed over 30 years ago that there are many exponential rigs satisfying the Alonzo Church-inspired condition that for any x such that $1 \leqslant x$, there exist y such that $x \leqslant y$ and $y^y = y$. Indeed, he actually proved the much better result that such rigs can arise objectively, i.e. by abstraction from suitable cartesian closed categories.

3. Amazingly tiny objects

There are still other relevant kinds of right adjoints, to which formal methods might be applicable. Two of these arise in connection with differential calculus; one involves representing the tangent-bundle functor, which is the most basic functor on smooth spaces. Since the object that represents a given functor is unique, the "amazingly tiny" object T that represents the tangent bundle of X as the exponential X^T deserves close study. It has the following rare property: one of the key virtues of objectifying higher types (i.e. of working in a cartesian closed category) is of course that we can treat as mere maps the "functionals of arity A from X to Y''

$$X^A \to Y$$

and hence compose these with other maps on either side; but in very special cases, such as $A = T$, the same information in those functionals can be represented as functions on X itself (with values in a more richly structured object)

$$X \to Y^{1/A}.$$

This possibility does not seem to have been contemplated by combinatory logic; the formalism should be extended to enable treatment of so basic a situation. Note that the existence of this exceptional right adjoint implies that $(\ )^T$ itself is the inverse image for a geometric morphism and hence for any positive structure carried by an object R, R^T will have the same structure. (However, R^T will not necessarily share nonpositive properties (i.e. the properties whose expression requires universal quantification or implication within the hypothesis of an inference) with R, contrary to the postulate of nonstandard analysis.)

4. Central idempotents

Another additional right adjoint exemplified by differential calculus is actually also left adjoint and indeed to the same functor. More precisely, a cartesian closed category might have a "central idempotent", i.e. a natural transformation θ from the identity functor to itself, each instance of which is idempotent. Then the subcategory of "θ-discrete" objects where this idempotent is the identity endomap will be both reflective and coreflective; in a broader context we might speak of "connected components" and "points" of any object as given by two special functors into the subcategory, but here these two are isomorphic. How can such a situation arise?

For example, if we divest smooth spaces of all global cohesion, keeping only the jets (on which the Thom–Mather singularities depend), we obtain a category in which every connected component of any object has exactly one point, so that the natural map between those two functors is an isomorphism. Of course, the result of splitting a central idempotent does usually not give a “fully discrete” cartesian closed category (i.e. abstract sets); for example, in infinitesimal algebraic geometry (over a non-algebraically closed field K of characteristic zero, i.e. presheaves on the opposite of the category of finite-dimensional local K-algebras) the nature of the point inside an infinitesimal component varies in the way studied by Galois.

Another very important example is the homotopy category constructed by Hurewicz. It is again a cartesian closed category carrying a central idempotent because the basic construction involves taking the path-components of an arbitrary space and considering them as “points” of the homotopy type of the space, and (due to the self-transitive nature of the continuous interval which parameterizes homotopies) this “points” functor not only remains left adjoint but becomes right adjoint too, to the same inclusion.

There is evidently a large, mostly unexplored, area concerning the cartesian closed categories that can be mapped to or from the spatial ones but that enjoy such further simultaneous right adjoints; although again this possibility does not seem to have been contemplated by combinatory logic, some of those adjoints are relevant to data transformation and to the qualitative features of the data that we need to transform.

5. Functional analysis versus “natural” numbers

Contrary to the uniqueness suggested by the term, “the” λ-calculus, there is still a vast variety of cartesian closed categories to be understood. In a topos we can construct the exponential spaces as in set theory from the power types. The power types also permit, by the Dedekind–Freyd infinite intersection method, the construction of natural number objects (and other left adjoints, such as minimal list models) given only the geometrical datum of a non-Dedekind-finite space. But such a geometrical datum could certainly belong to a cartesian closed subcategory which does not contain that extreme subobject. More generally, any category with products, for example a model of O-minimal analysis, can be embedded in a cartesian-closed category. Indeed from [9] one can see that both

(1) the intrinsic meaning given by Volterra and his students to “analytic functionals”, like the Hadamard definition of multi-dimensional smoothness rediscovered by Boman and extensively used by Frölicher, Kriegl, and Michor in infinite-dimensional smooth analysis [6], and

(2) the Banach–Mazur notion of recursive functional, later further developed by Ersov and by Mulry [12] (see also [2]), are essentially contained in the Yoneda–Cayley–Dedekind tautological method of obtaining cartesian-closed embeddings. Of course, one virtue of that method is that the category obtained is actually a topos, but if one wished to avoid the power types, one could simply take exponentials of exponentials.... of representables inside such a topos.

There is a philosophical interpretation of the Dedekind–Freyd construction: The “mild” objectification of the subjective (involved in the postulation of a truth-value object) almost implies (if we use its full universal property) the stronger objectification of the "completed" subjective process (i.e. the bad infinity that the O-minimal theory of the continuum manages to avoid). Can O-minimal models be embedded into cartesian-closed categories without introducing the left adjoints like the Dedekind–Lawvere N? Since that question probably does not have a simple yes-or-no answer, I will consider it in some other aspects below.

Given any commutative ring object R we can always construe $R^2 = R[i]$ as a new ring with complex multiplication, inside of which there is the multiplicative subgroup S^1 corresponding to the circle. If we are in a cartesian-closed category, we can form the subspace $Z = \mathrm{Hom}(S^1, S^1)$ of $(S^1)(S^1)$ consisting of homomorphisms; since S^1 is an abelian group, Z is a ring. That ring is usually considered to be "the integers" when R is in some sense a one-dimensional continuum. However, there seems to be nothing in this entirely right-adjoint construction to imply that Z satisfies any left adjoint sort of universality such as a suitable version of Dedekind universal recursion. For example, Moerdijk and Reyes [11] find that the Z arising from the smooth reals (the Poincaré group of S^1 and hence the appropriate coefficient group for homology in a smooth category) is not the usual discrete $N[-1]$ but rather has itself some residual smoothness. Which automorphisms of objects are tame enough to admit iteration parameterized by Z? Does the existence of Z or similar objects entail in some indirect way the existence of N?

Because the category $\mathscr{S}$ of abstract sets is not a universal arena, but merely a useful auxiliary tool for the study of continuous space and motion, it is possible to separate the roles of objective and subjective infinity even more consciously than Cantor and Dedekind did. A topos $\mathscr{T}$ can be generated over $\mathscr{S}$ by a geometrical subcategory $\mathscr{C}$ which idealizes the objective infinity of time and space intervals; $\mathscr{T}$ will then also contain the idealizations N and Ω of subjective infinity which "objectify" recursion and truth. However, N and Ω need not belong to $\mathscr{C}$, as shown by Tarski's real semi-algebraic geometry and its later developments which study $\mathscr{T} \to \mathrm{O}$ as extensions of a known decidable theory O. Note that $N \subseteq R \in \mathscr{C}$ does not imply $N \in \mathscr{C}$; even though any subobject of R is an equalizer in $\mathscr{T}$, it may not be possible to find an equation $R \rightrightarrows C$ defining it in $\mathscr{C}$ itself. The striking geometric property of $\mathscr{C}$ is that every space in it has only finitely many connected components, even though $\mathscr{C}$ is closed under equalizers. Thus the question becomes: "How much analysis can be done in $\mathscr{C}$?" (Here we refer, of course, to analytic geometry, differential equations, etc., rather than to realms such as abstract measure theory whose very foundations in the 20th century became so deeply intertwined with the "completion" of subjective infinity.) The line of investigation based on the idea that a one-dimensional part $\mathscr{T}_1$ of $\mathscr{T}$ is equivalent to O has led to richer and richer $\mathscr{C}$. The question posed here is: "Can we tame [5] some form of functional analysis as well, i.e. can $\mathscr{C}$ be cartesian closed?"

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