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FOUNDATIONS AND APPLICATIONS: AXIOMATIZATION AND EDUCATION

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Abstract. Foundations and Applications depend ultimately for their existence on each other. The main links between them are education and the axiomatic method. Those links can be strengthened with the help of a categorical method which was concentrated forty years ago by Cartier, Grothendieck, Isbell, Kan, and Yoneda. I extended that method to extract some essential features of the category of categories in 1965, and I apply it here in section 3 to sketch a similar foundation within the smooth categories which provide the setting for the mathematics of change. The possibility that other methods may be needed to clarify a contradiction introduced by Cantor, now embedded in mathematical practice, is discussed in section 5.

In my own education I was fortunate to have two teachers who used the term "foundations" in a common-sense way (rather than in the speculative way of the Bolzano-Frege-Peano-Russell tradition). This way is exemplified by their work in *Foundations of Algebraic Topology*, published in 1952 by Eilenberg (with Steenrod), and *The Mechanical Foundations of Elasticity and Fluid Mechanics*, published in the same year by Truesdell. The orientation of these works seemed to be "concentrate the essence of practice and in turn use the result to guide practice". I propose to apply the tool of categorical logic to further develop that inspiration.

Foundations is derived from applications by unification and concentration, in other words, by the *axiomatic method.* Applications are guided by foundations which have been learned through education.

Since the most fundamental social purpose of philosophy is to guide education and since mathematics is one of the pillars of education, accordingly philosophers often speculate about mathematics. But a less speculative philosophy based on the actual practice of mathematical theorizing should ultimately become one of the important guides to mathematics education.

§1. Axiomatically reconstituting geometric objects as structures. The axiomatic method involves concentrating key features of ongoing applications. For example, Cantor concentrated the concept of isomorphism, which he had extracted from the work of Jakob Steiner on algebraic geometry. The indispensable "universe of discourse" principle was refined into the idea of

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structure carried by an abstract set, thus making long chains of reasoning more reliable by approaching the ideal that "there is nothing in the conclusion that is not in the premise". That vision was developed by Dedekind, Hausdorff, Frechet, and others into 20th century mathematics.

The connection of Cantor with Steiner is not mentioned in most books (there is an unfortunate tendency for standard works on the history of science to perpetuate standard myths, rather than to discover and clarify conceptual analyses). In fact, Cantor wrote in 1882 (*Mathematische Annalen*, Band 20): "Den Ausdruck 'Mächtigkeit' habe ich J. Steiner entlehnt." Steiner had used it in "einem verwandten Sinn" involving the existence of bijections between Mengen having projective structure and preserving that structure; however, Cantor instead wants "keine Beschränkung auf Stetigkeit und Unstetigkeit" in his bijections. This conscious negation of the cohesive Mengen of geometry and analysis was of course motivated by problems involving trigonometric series; it was repeated in the famous "Beiträge zur Begründung der transfiniten Mengenlehre" of 1895 (*Mathematische Annalen*, Band 46); he emphasized that it should apply also to arithmetic, function theory and other parts of geometry. Again, the standard works in the 20th century said little about Cantor's resulting contrast between cohesive Mengen of the various categories and the structureless sets or "Kardinalzahlen", even though it would seem to be a fundamental issue for foundations to give an exact account of how the Mengen are to be reconstituted as structures on the structureless sets. Note that his 1895 description of the precise relation between the points in a given Menge and the corresponding "lauter Einsen" in its Kardinalzahl shows that Cantor was not using the term "Zahl" in the sense of an isomorphism-class or a canonical representative thereof.

The above-mentioned goal, of ensuring that logical conclusions are not contaminated by the background, is essentially achieved for first order model theory by assuming that structureless sets satisfy the axiom of choice (epimorphisms split) since that implies the Nullstellensätze required for the completeness theorem; the achievement of this goal is less clear for model theory of the second order theories which are usually considered to be necessary for modeling cohesive spaces.

Even though in everyday practice the axiom of choice is the basic distinguishing feature of categories of structureless sets, they cannot be completely described by any finite or recursive list of axioms; despite Cantor's apparent belief in their absoluteness, Gödel's incompleteness result applies here as much as it does to arithmetic.

But are not abstract sets themselves problematic? Does not perhaps the very idea of them involve a contradiction? In considering that question one should not confuse contradiction and inconsistency. A. Tarski defined in the 1930s an "inconsistent" formal system to be one in which everything is provable (clearly such a system is useless); by contrast, whether A and not-A

can co-exist depends on the precise rules of inference and meanings that the system attributes to negation. One of the principles of dialectics is that specific contradictions are the key to all development. For example, complex analysis had needed to become explicit on a notion which is fundamental to general topology: in the category of closed subspaces of a space, (A and not-A) $=$ the *boundary* of A, as is made precise in the formal system of co-Heyting algebra. (To move from a room A into not-A one must pass through the threshold, at least if motion is parameterized by a time interval which is a connected space.)

Yes, the notion of an abstract set (Cantor's Kardinalzahl) is a strong contradiction: its points are completely distinct and yet indistinguishable. How is this contradiction resolved in mathematical practice; indeed where do the abstract sets come from? Cantor extracted his notion of abstract set from the ongoing mathematics. Over a century later, we can describe the process of extraction more explicitly, as will be sketched below.

Several branches of ongoing mathematics, related to algebraic topology and functional analysis, can be viewed as taking place in categories of the sort to be described; examples are the category of categories and the category of smooth spaces. Within each such category we will isolate a relatively discrete subcategory; and having isolated a relatively discrete subcategory of spaces, we can proceed to the study of a general space X in the category in terms of its way of contrasting a particular concentration T of cohesion with the discrete, essentially by taking the discrete set of points of the function space X^T, on which act the incidence relations named by self-maps of T. The use of this contrast as a method became explicit around 1960.

§**2. Systems of variable quantities need domains.** Variable quantities occur in all applications of mathematics. Therefore, a useable description of the concept of variable quantity and of its main features must be a key part of foundations. A first step in the dialectical analysis of the concept is to resolve it into space versus quantity, so that "spaces" serve as definite "domains of variation" for the variable quantities. There are many categories of spaces (smooth, combinatorial, etc.) and many kinds of quantity thereon (smooth or semi-continuous real quantities, truth values, etc.). Extensive versus intensive is another fundamental dialectic arising within variable quantity; both of those modes of variation abound, although that fact is somewhat obscured if one attempts to "reduce" extensive quantities to intensive quantities. From a philosophical or physical point of view it is perhaps the extensive quantities which are more basic, since they include mass, volume, charge, energy, entropy.

If E is any covariant functor on a category $\mathcal{X}$ of spaces, then knowing only that $\mathcal{X}$ has a terminal object 1, we can define three ideas basic to the interpretation of E as a type of extensive quantity:

(E1) The unique morphism $X \to 1$ on any space X in $\mathcal{X}$ induces a map (in the codomain category) $E(X) \to E(1)$ which assigns to each E-extensive quantity on X its total. This induced map deserves to be called the *integral* over X (relative to E).

(E2) For any given point $x: 1 \to X$ of a space X there is the induced map $E(x): E(1) \to E(X)$ which gives an extensive quantity supported on x having any given total. Thus that induced map deserves to be called the *Dirac delta* at x (relative to E).

(E3) A given extensive quantity μ on X is *supported* on $a: A \to X$ (extending the notion of an element belonging to a subobject) if there exists an E-quantity ν on A which $E(a)$ carries to μ.

Dually, for any contravariant functor F defined on a category $\mathcal{X}$ with terminal object, we can define three ideas which are basic to an interpretation of F as a type of intensive quantities:

(F1) The unique $X \to 1$ induces $F(1) \to F(X)$ which is the *inclusion* of *constant quantities* as special variable ones.

(F2) Any point $x: 1 \to X$ induces $F(x): F(X) \to F(1)$ which is the *evaluation* at x of any F-variable quantity on X (interpreting the result as a constant quantity).

(F3) Given any $a: A \to X$ there is the important extension problem: Given f in $F(A)$ it is *extendible* along a if there exists g in $F(X)$ which is carried by $F(a)$ to f.

Commonly-analyzed examples of extensive quantity-types arise by the Riesz paradigm $E(X) = Hom(F(X), E(1))$, where F is some given contravariant functor playing the role of intensive quantities. For example, if F consists of smooth functions (or differential forms), the corresponding Riesz-deRham extensives are the usual distributions (or currents) of compact support valued in $E(1)$.

Also very basic is a further determination of intensive quantities as possible *ratios* or *densities* between given extensive quantities. ("Possible ratio" means a homomorphism, which is only occasionally determined uniquely by a given input (or divisor) and output (or dividend); indeed a ratio may not even exist for given divisor and dividend; the condition for such existence is absolute continuity.) Thus intensive quantities should act $E_1 \to E_2$ (between possibly different extensive types), but more precisely, there is the strong naturality requirement that is sometimes called the projection formula in algebraic topology: for each space X there is the category $\mathcal{X}/X$ of spaces over X, with its forgetful functor to $\mathcal{X}$; composing E_1 and E_2 with that forgetful functor, we get two functors on $\mathcal{X}/X$, and the space of natural transformations between these may be called $F_{12}(X)$. If $X' \to X$ is any map in the category $\mathcal{X}$ of spaces, there is the functor $\mathcal{X}/X' \to \mathcal{X}/X$ obtained by composing; restriction along that functor gives contravariantly a map $F_{12}(X) \to F_{12}(X')$. Not only contravariance, but multiplicative structure, is characteristic of intensive quantities; in our semi-explicit example the multiplication

$$F_{12}(X) \times F_{23}(X) \to F_{13}(X)$$

is just given by composing natural transformations and it is preserved by change of space. The usual integral of an intensive f with respect to an extensive μ is the total of the result $f \cdot \mu$.

The "restriction" functorality, which applies to intensive quantities and gives rise to the extensibility problem, has in general no meaning for extensive quantities. On the other hand, the "push-forward" functorality, which gives rise to the support problem, has sense for extensive quantities, but not in general for intensive ones.

§3. **Geometry provides its own foundation.** Commonly, the codomain category for a quantitative functor E or F on $\mathcal{X}$ is a category $\mathrm{Mod}(\mathcal{X})$ of linear structures in $\mathcal{X}$ itself; thus it is most basically the nature of the categories $\mathcal{X}$ of spaces (that such systems of quantities have as domain) which needs to be clarified. Concentrating the contributions of Volterra, Hadamard, Fox, Hurewicz and other pioneers, we arrive at the important general idea that such categories should be *cartesian closed*. For example, the power-set axiom for sets is one manifestation of this idea—note that it is not "justified" by the 20th century set-theoretic paraphernalia of ordinal iteration, formulas, etc., since it (together with the axiom of infinity) must be in addition assumed outright. Indeed, often the needed categories of spaces are self-founded in the sense that within them a subcategory playing the role of abstract structureless sets can be defined, and the general space in the category can be reconstituted in terms of specific structure, with the general meaning of such structures also concentrated into some particular objects. It turns out that this "geometrical" mode of analysis (i.e., essentially the analysis of a general object in terms of incidence relations between figures of special shape) applies not only to smooth, analytic, and algebraic spaces, but also to the category of categories. The latter is itself a cartesian-closed category in which there is a particular figure-shape $T =$ "the generic arrow" in terms of which a general category can be reconstructed from a diagram of discrete ones.

If a functor is representable, its representing object is unique (as a consequence of the Cayley-Dedekind-Yoneda lemma). The most basic functor in multi-dimensional differential calculus is the tangent bundle. Thus in each determination (smooth, analytic, or algebraic) of a suitable category of spaces, there is a definite space T such that X^T is the tangent bundle of the arbitrary space X. The existence of this space T is very distressing to those who (like Berkeley and Bolzano?) would prefer that the foundational insight "curved paths become straight in the infinitely small" not be applied. But if we actually look at T, we find that it has a host of remarkable properties. In particular, there are several senses in which the tiny T generates a whole category $\mathcal{X}$, including infinite-dimensional spaces; for example, the usual finite-dimensional varieties are equational subspaces of the spaces $(T^T)^n$. The multiplication of real ratios may be *defined* to be the composition of endomaps of T ("speedups"), leading to a realistic "foundation" for real analysis.

The space T is often described in terms of the algebra of functions on it, wherein every quantity is a sum of a constant and a quantity of square zero, multiplied by the Leibniz product rule (Leibniz apparently only hinted at this space; of course, like the complex numbers which he recognized as partly analogous, the algebra is not an elementary extension of the constants. A century ago E. Study urged explicit recognition of these matters, and the work of Kähler a half century ago led to their incorporation in Grothendieck's algebraic geometry). In fact, the foregoing description applies equally in algebraic, analytic, and smooth geometry, showing that the uniqueness of T is even stronger than the usual uniqueness within a given category.

In the symmetric power $S = T^2/2!$ the two "axes" are identified with T, so that for each space X there is a restriction map

$$X^S \to X^T$$

from infinitesimal quadratic paths. A given section (= "prolongation operator") for this restriction is a second-order differential equation with configuration space X; the category whose objects are spaces equipped with such ODE's (and whose morphisms are lawful motions) is also cartesian closed [see Lawvere 1997], sharing many of the good properties of the original category of spaces.

Objects such as the T described above are given by concentrating applications; their content is essentially the *kind* of cohesion and variation that typically arises in using the spaces and maps in their category. Given such objects T, a contrasting category of spaces *lacking* cohesion and variation can be defined to consist of objects X equipped with prolongation operators along the map $T \to 1$ to the terminal object (rather than to the symmetric power as in the previous paragraph); in such an X, all T-parameterized paths are constant. In case the terminal map is epic these discrete rigid spaces form a full subcategory of the original, but even without that assumption, they will form a cartesian-closed category $\mathcal{S}$ with very good properties. To the extent that T itself is not discrete and rigid, the physically relevant real analysis associated with the monoid T^T is available; for example, the tangent bundle representor T has just one point $1 \to T$, but its own tangent bundle $T^T \neq T$ since it has at least two distinct points 0 and 1.

The inclusion functor of the discrete rigid spaces into the category of all spaces of the given kind typically has both left and right adjoints

$$\mathcal{X} \underset{\pi_0}{\overset{\coprod}{\longrightarrow}} \mathcal{S}$$

The left adjoint π_0 provides the basis for the qualitative homotopical analysis of the general spaces since, as Hurewicz made explicit, the definition $[X, Y] = \pi_0(Y^X)$ determines a new category $Ho(\mathcal{X})$ still based in $\mathcal{S}$. In fact, this new homotopical category is itself cartesian closed and often has the remarkable property that the right adjoint to the discrete inclusion also serves as the left adjoint to the same ("points" = connected components).

The right adjoint to the discrete inclusion is essentially Cantor's abstraction process which assigns to every cohesive "Menge" X its set $|X|$ (or "Kardinalzahl") of bare points. But because $\mathcal{X}$ is cartesian closed, the abstraction process also assigns to X the set $|X^T|$ of figures of shape T in X. Moreover, the abstract sets $|T^S|$ act on these figures by means of maps $|X^T| \to |X^S|$, in a way compatible with composition $|T^S| \times |U^T| \to |U^S|$, since the right adjoint preserves products. If we select a few objects $T, S, \ldots,$ (for example the first four ordinals in the case of the category of categories, or the infinitesimal and short path-parameterizers in the case of smooth spaces), then the map sets $|T^S|$ will determine an internal category $\mathbf{T}$ in $\mathcal{S}$ and the analysis of a space in terms of **T**-figures and incidence relations becomes a functor

$$\mathcal{X} \to \mathcal{S}^{\mathbf{T}^{op}}$$

to the category of internal presheaves on **T**; that functor typically has a left adjoint which "presents" some of the spaces by gluing the selected basic figures in various ways. (We concentrate on the situations where **T** consists of spaces constructed from those T which helped to define $\mathcal{S}$.) The "adequacy comonad" on $\mathcal{X}$ is the endofunctor defined by composing this "geometric realization" functor with the figure-and-incidence analysis; the natural map from that composite to the identity will be an isomorphism in case an adequate **T** has been chosen.

Usually, the points functor has a still further adjoint, "codiscrete", which together with the discrete inclusion constitutes a pair of unified opposites; with any space, two very special spaces with the same points are thus associated, one with maximal distinctness (hence no motion), the other with maximal unity (and hence no distinguishing properties). In that way the contradictory aspects of Cantor's Kardinalzahl can be usefully displayed, in their role as a zeroth step in the dialectical analysis of any space in terms of appropriate skeleta and co-skeleta.

The above mode of analysis of a category $\mathcal{X}$ was essentially displayed around 1960 by Cartier, Grothendieck, Isbell, Kan, and Yoneda, and has been widely used since. The less-widely-known fact that the discrete rigid "foundation" $\mathcal{S}$ can be defined from within $\mathcal{X}$ was first exploited in my thesis, for the case of the category of categories.

§**4. How different are cohesive and abstract sets?** To what extent does the above analysis realize Cantor's abstraction to structureless sets? Of course, if we took **T** to consist only of the terminal space 1, the whole analysis would reduce to the identity, whereas an aim of the program is to understand important spaces in terms of objects which have simplifying properties that the important spaces themselves could not possibly have, such as the axiom of choice (epimorphisms split), or at least its consequence, the law of excluded middle. But the relative nature of the construction suggests that we may not always want to go all the way; for example, if $\mathcal{X}$ embodies algebraic geometry over a non-algebraically-closed field, it is appropriate to modify **T** so that the objects in the resulting $\mathcal{S}$ retain at least the structure of Galois-group action. One striking expression of the contrast between $\mathcal{X}$ and $\mathcal{S}$ often occurs when $\mathcal{X}$ is a topos and π_0 of its truth-value object is 1 in $\mathcal{S}$; this connecting of true to false will occur whenever the real-ratio object T^T is connected, as Grothendieck pointed out in 1983.

Of course the properties, that the category $\mathcal{S}$ of relatively discrete objects has, will depend on axiomatically given properties of $\mathcal{X}$, only some of which are involved in the possibility of an appropriate choice of a category **T** of basic figures. Part of what needs to be made explicit are the conceptual relationships (here for example the unity and identity of adjoint opposites), about which properties speak. An inadequate legacy is the one that attached the term "Begriffschrift" to a calculus of mere properties, incorrectly suggesting that concepts can be adequately characterized in terms of properties alone.

§**5. How constant is constant?** Having introduced the structureless sets into modern mathematics, Cantor had conjectured that they satisfy his continuum hypothesis. Essentially, that conjecture was proved later by Gödel. Since Gödel's result has been underestimated by many people (including Gödel himself) as a mere relative consistency result, I will sketch below what seems to be its relationship with Cantor's contrast between structureless sets and cohesive Mengen. Since Cantor himself made the conjecture shortly after he had conceptualized the distinction between Mengen and Kardinalzahlen, and since modern mathematics has revealed, in the vast diversity of mathematical categories of cohesive and variable Mengen, that it is absurd to ask whether the continuum hypothesis as such is "true" without a guiding concept, it is reasonable to interpret Cantor's conjecture as:

for the topos $\mathcal{E}$ of structureless sets, CH is true in $\mathcal{E}$.

However, Gödel's incompleteness theorem shows that we cannot fully characterize "structureless", except in a relative sense. Thus we are led to modify the above implication to:

for a topos $\mathcal{E}$ of sufficiently structureless sets, CH is true in $\mathcal{E}$.

It is this quantitative version which Gödel's result supports.

Such a scale of relative inertness vs. activity still needs to be clarified. We consider that the points on the scale correspond to certain toposes, because a topos is just any category closed under the internally-characterized function-space and power-set functors. The toposes of Mengen encountered in mathematics have conceptual features which we call cohesion and variation, each of which may occur in relatively pure form. Even though the cohesive toposes are the ones typically appropriate for the relative foundation discussed in section 3, I will refer to the feature of categories which model Cantor's negation of Mengen as "constancy", because the merely logical features of a topos can always be entirely reflected in a (non-unique) topos of the purely variable kind. More precisely, a result of Deligne, refined by Freyd, Johnstone, Joyal, Tierney, and Moerdijk, replaces an adequate internal category **T** of geometric figures by the category $F(\mathbf{T})$ with the same objects, but with formal strings of arrows from **T** as its morphisms, and uses this to construct a new topos $F(\mathcal{E})$ based on the same $\mathcal{S}$ as $\mathcal{E}$. Because all morphisms within $F(\mathbf{T})$ are epimorphisms, $F(\mathcal{E})$ looks merely variable to $\mathcal{S}$; but the careful construction leads to a geometric morphism $F(\mathcal{E}) \to \mathcal{E}$ whose inverse preserves the internal logic (as well as the external cohomology, even though the coherence of the incidence relations between figures required of the spaces in $\mathcal{E}$ has been largely relaxed).

Most of the obviously variable categories do not satisfy CH: a variable set might be countable for a while and continuum-sized for the rest of the time, and clearly such a set would lie strictly between the constant extremes. The construction by Cohen and others, of categories $\mathcal{E}$ satisfying the previously-known axioms for constancy (implied by ZF), but not satisfying CH, is essentially achieved by first constructing a suitable category of strictly variable sets and then "freezing", not at an actual point of the parameter domain, but at a point of its compactification.

The ensuing decades of work by set theorists have shown that quite a variety of positionings of power sets within the ordering of sizes is compatible with the standard axioms; thus these standard axioms are very far from insuring constancy, while on the other hand they were implicitly designed to describe constancy (note that very elementary properties, such as "any two maps from A to B are distinguished by some point of A," and "any two points of A are exchanged by some automorphism of A" are deeply intertwined with the conceptual basis of these axioms, so that they do not apply directly to the cohesive and variable sets of mathematics.)

Now, in the practice of geometry, analysis, model theory, combinatorics, etc., the toposes of Mengen that arise are typically related to one another by "geometric morphisms" which are actually adjoint pairs of left exact functors; it is in these terms that mathematical comparisons can be made, the functors in both directions serving to establish a kind of commensurability. In particular, the relation of $\mathcal{E}$ to a base topos $\mathcal{S}$ is a given geometric morphism M from $\mathcal{E}$ to $\mathcal{S}$ (just as in algebraic geometry spaces are "based" on *spec*(K) for some ring K); even though it is important to work over general bases $\mathcal{S}$, for our purposes here we want to consider that $\mathcal{S}$ is "constant", which means that it at least satisfies the axiom of choice. Using the right adjoint to define $\mathcal{E}(X, Y) = M_*(Y^X)$ we can view the "hom sets" of $\mathcal{E}$ as objects of $\mathcal{S}$, whereas the left adjoint M^* not only includes $\mathcal{S}$ as the discrete objects of $\mathcal{E}$, but permits many apparently infinitary constructions in $\mathcal{E}$ to be expressed in finite terms. The comparison of toposes by means of geometric morphisms fits nicely with the mathematical program sketched in section 3 for finding constancy/inertness as a full subcategory of variation/cohesion, because that program indeed produces adjoint pairs.

To compare degrees of constancy we apparently need to consider comparisons involving only a single non-full functor, rather than an adjoint pair. Such comparisons were considered by Gödel around 1930 and from a categorical point of view involve at least the following fragments of Gödel's notion of "absolute properties": If I is a product-preserving functor, then for two objects A, B, in the domain, there is a comparison map in the codomain

$$I(B^A) \to (IB)^{(IA)}$$

but there may in general be no reverse interpretation of map spaces in the codomain category in terms of objects in the domain category. In case the domain category is the more constant, then the comparison map is injective and may be interpreted as the inclusion of the maps which are more inert into those less inert. On the other hand if I is the "freezing" process used to prove independence results by deriving a pseudo-constant codomain category from an appropriately variable domain, the comparison map is actually an isomorphism.

The following engineering analogy may be suggestive: As I learned in my days as a cyclotron technician, in order to more closely approach the ideal of a perfect vacuum, it is often necessary to employ machines that use several distinct technologies in succession, even though ordinary vacuum pumps are sufficient for everyday needs. After a conventional pump has evacuated as much air as it can, another more sophisticated device using oil drops takes over to further reduce the pressure, and sometimes even a third technology is needed. (Attempts to approach a temperature near absolute zero may also involve successive, qualitatively more extreme, techniques of cooling). Similarly, if we need to approach Cantor's ideal of zero cohesion and variation, everyday techniques may need to be supplemented with more sophisticated methods; instead of using pressure and temperature "gauges", we "see" what degree of constancy has been achieved in terms of special properties of the categories.

Any set theory worthy of the name permits a definition of mapping, domain, codomain, and composition; it was in terms of those notions that

Dedekind and later mathematicians expressed structures of interest. Thus, any model of such a theory gives rise to a category and whatever complicated additional features may have been contemplated by the theory, not only common mathematical properties, but also most interesting "set theoretical" properties, such as the generalized continuum hypothesis, Dedekind finiteness, the existence of inaccessible or Ulam cardinals, etc. depend only on this mere category.

Now Gödel's "constructible" sets L in themselves have a tremendous amount of structure, but the relevant category derived from them is, remarkably, in general much more constant (structureless!) than that devised from an original ambient model V.

GCH is weaker than $V = L$; extensions like $L[\mu]$ of L may also satisfy GCH, even if they are not "as" constant as L. But in the other direction, L itself is probably not the ultimate vacuum technology. The uniqueness of L, (like that of the natural numbers) is only relative to a given category V. Such uniqueness tends to persist when we change categories along adjoint pairs of functors. On the other hand, incompleteness theorems and nonstandard model constructions show that with unidirectional functorial changes, there is always the possibility of finding additional classes that should have been involved in that Dedekind big intersection which aimed toward "the" least such and such. This sort of undermining of uniqueness can apparently succeed even while preserving a given finite set of properties (for example, the axioms for Gödel-Bernays class theory together with $V = L$). These considerations, among others, suggest that if indeed investigations of the constancy ideal itself, not only of its useful contrast with cohesion and variability, are to be pursued, strengthenings (as well as weakenings) of L may need to be considered.

During the past forty years we have become accustomed to the fact that foundations are relative, not absolute. As section 5 suggests, I believe that even greater clarifications of foundations will be achieved by consciously applying a concentration of applications from geometry and analysis, that is, by pursuing the close dialectical relation between foundations and applications. The pursuit of the relation between a topos of Mengen and its possible self-foundations, as outlined in section 3, has still many unexplored but geometrically relevant roads to follow.

REFERENCES

[1] P. Cartier, *Groupes algébriques et groupes formels*, ***Colloq. théorie des groupes algébriques, (Bruxelles 1962)*** (Paris), Gauthier-Villars, pp. 86–111.

[2] S. Eilenberg and N. Steenrod, ***Foundations of algebraic topology***, Princeton University Press, 1952.

[3] A. Grothendieck, *Techniques of construction in analytic geometry*, ***Cartan seminar***, 1960.

[4] W. HUREWICZ, *k-spaces*, unpublished lecture referred to by David Gale, *Compact Sets of Functions and Function Rings*, ***Proceedings of the American Mathematical Society***, vol. 1, 1950, pp. 303–308.

[5] J. ISBELL, *Adequate subcategories*, ***Illinois Journal of Mathematics***, vol. 4 (1960), pp. 541–552.

[6] F. W. LAWVERE, *Thesis: Functorial semantics of algebraic theories, Columbia University*, 1963, (reprinted Buffalo Workshop Press, POB 171, Buffalo, N.Y. 14226).

[7] ———, *The category of categories as a foundation for mathematics*, ***La Jolla conference on categorical algebra***, Springer-Verlag, 1966, pp. 1–20.

[8] ———, *Categories of space and of quantity*, ***The space of mathematics: Philosophical, epistemological and historical explorations***, De Gruyter, Berlin, 1992, pp. 14–30.

[9] ———, *Cohesive toposes and Cantor's "lauter Einsen"*, ***Philosophia Mathematica***, vol. 2 (1994), pp. 5–15.

[10] ———, *Toposes of laws of motion*, 1997, video transcript:
`http://www.acsu.buffalo.edu/~wlawvere/downloadlist.html`.

[11] ———, *Volterra's functionals and the covariant cohesion of space, Categorical studies in Italy*, ***Rendiconti del Circolo Matematico di Palermo. Serie II. Supplemento***, vol. 64 (2000), pp. 201–214.

[12] C. TRUESDELL, *Mechanical foundations of elasticity and fluid mechanics*, ***Journal of Rational Mechanics and Analysis***, vol. 1 (1952), pp. 125–171 & 173–300.

[13] V. VOLTERRA, *Sur une généralisation de la théorie des fonctions d'une variable imaginaire*, ***Acta Mathematica***, vol. 12 (1889), pp. 233–286.

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