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Categorical algebra for continuum micro physics

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Abstract

Using the setting of a topos equipped with a specified infinitesimal time-interval, we try in part I to clarify the idea of lawful motions as morphisms in a category whose objects are laws of motion on state spaces, and in part II to develop specific relations between states, bodies and particles. A very general scheme to make mass distributions yield notions of inertia and hence of force is discussed. Part III concerns a special notion of a body having just one point, yet containing rich microstructure; such a body is placed in space and, in general, treated like any other body. In part IV some detailed homogeneous and quadratic examples are defined.

MSC: 18A40; 18B25; 34G99; 58A03; 74A60

1. Introduction

Why should categorical algebra serve as an appropriate and useful guide to understanding and relating the vast variety of continuum models which arise in engineering science? Very basically, it is because that variety is of two kinds: there are many models within a given category and there are also many related categories, and moreover, the necessary motion of thought in those two dimensions requires careful algebraic calculation, not mere speculation. Categorical algebra is most fundamentally the study of 2-categories, which typically objectify situations like that just described, and which provide the natural setting for that powerful tool of concept-formation known as adjointness. For our purpose, we need to consider both categorical algebra in general and also certain toposes conditioned by specific infinitesimal adjoints.

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The two basic Steenrod division problem forms, lifting and extending, proliferate in a general 2-category into many problem forms, especially if one assumes that some given 1-cells have adjoints, or that some previously stated problems have given solutions. Geometric objectification, that is, the careful introduction of "spaces" as objects to represent the concepts involved, is used to bring this complex dialectic down to earth, in a way explicit enough to be used as a guiding framework for formulating and developing ideas by clarifying the contrasts—becoming vs. being—body vs. particle—motion vs. law—solution vs. problem. In particular, by introducing spaces into the algebra we can objectify the comparison of constant and variable (i.e. measurement). Hence, in verifying the effectiveness of the guide, the needed propositions are about the fundamental forms of functional analysis and differential equations, because notions such as space of quantities, approximation, and internal cause, become definable.

I have proposed that any infinitesimally generated topos of spaces is a useful framework for algebraically interpreting Galileo's *Two New Sciences*, in a way that includes infinite-dimensional systems such as elasticity and electromagnetism [6,7]. These two new sciences are essentially the science of dynamics and the science of materials:

- In the science of dynamics, laws of becoming are deemed to act on states of becoming and not merely on configurations of being,
- in the science of materials, configurations are resolved into relations between body and space and corresponding states can be defined; then laws of becoming for the states are constitutively conditioned by those configurational relations.

In my talks at the 1993 Hamilton sesquicentennial meeting in Dublin and at La Sapienza in Rome in 1995, I used the term "Algebra of Time" to refer to certain consequences of the above dynamical principle, especially to the fact that the dynamical time which parameterizes lawful motions is not one-dimensional. I will make precise one of the ways in which that fact can be expressed; namely the usual functor, which derives a first-order ODE on a state space from a higher-order ODE on a configuration space, has a left adjoint, which can be applied to a one-dimensional parameterizer in the first-order category. The categories that are related by these functors are actually toposes, as shown in my 1997 Montreal talk "Toposes of Laws of Motion" [6] and again in another way by Kock and Reyes [9].

Most categories are not toposes, but the categories arising in geometry and dynamics are often at least extensive and therefore have a useful full embedding into a cartesian closed environment which has very good exactness properties. The general axiom of extensivity merely requires that the category contain the possibility of non-cohesiveness; surprisingly, this seems to provide the appropriate general environment for the study of actual cohesiveness, and even for measuring it in terms of "higher connectivity". Thus that part of the accumulated knowledge of set-theoretical constructions, which remains when one relaxes the Cantorian abstractness and restores some cohesion to the "sets", is always available.

In several crucial cases, for example microlinearity and Frölicher duality, the category of those spaces satisfying some strong condition of importance is actually a reflective subcategory of a topos, with a reflector preserving finite products. Those topos objects that go beyond the special subcategories in principle serve to concentrate and objectify i.e. represent concepts based already on the special spaces, and hence permit at least partially the study of those concepts as such by the methods appropriate to spaces. Of course, it is crucial that the toposes can change into new toposes and back during the course of an investigation. Occasionally, it is useful to assume further conditions of completeness vs. smallness on the topos, for example, that the etale objects are reflective or that the locally separable objects are coreflective. In general, it is plausible to assume that some given functor has an adjoint if it does not appear to obstruct the hypotheses of the adjoint functor theorem, but it seems better not to make such assumptions in a wholesale manner since, for example, there is always the question of whether special properties are preserved by constructions of a new topos from a given one. As was worked out in the past century, many of the more "elementary" cases of this objectivizing principle follow from the existence of free monoids, i.e. from the existence of a natural numbers object. We use below a form of countable iteration a bit more general than free monoids, namely free *monads* generated by certain *generalized objects*. Note that a seemingly different form of countable iteration is involved in forming pullbacks of toposes bounded over a given base.

I

The basic setting is a topos $\mathscr{E}$ equipped with a specified pointed object T, an "amazingly tiny object" (ATO) in the sense that $()^T$ has a right adjoint $()^{1/T}$. [1]

T usually satisfies some very strong further properties. For example, the point $1 \xrightarrow{0} T$ is unique. The submonoid $R \subset T^T$, of those (parameterized) endomaps that fix the point, is commutative. In fact R usually has a well-determined addition as well and serves as a codomain for variable intensive quantities on arbitrary spaces. Actions of powers of the commutative monoid R are at the base of Birkhoff's theory of dimensional analysis. The object T is Dedekind-finite in the sense that the endomorphism monoid T^T, although not commutative, satisfies "xy invertible iff x invertible and y invertible". A property which is perhaps more directly suggestive of "tinyness" is that any monoid generated by the pointed object T is commutative; this property is directly related to the central example of second-order ODEs as reviewed below. An efficient

[1] Re: Terminology: In my 1980 Amiens lecture [Cahiers de Top. et Géom. Diff., XXI, 337–392], where I introduced these special objects into geometry, I expressed Grothendieck's exclamation point notation (for the unexpected right adjoint a geometric morphism $\mathscr{E}' \to \mathscr{E}$ may sometimes have) by the word "amazing"; it is especially surprising when the morphism is of the form $\mathscr{E}/T \to \mathscr{E}$. The same objects were called "tiny" by Freyd and Yetter [13] who proved some of their basic properties. In a presheaf topos the ATOs are the quarrable representable objects, surely small compared with the general presheaf, but there typically these representables are spaces like finite-dimensional Euclidean spaces. However, when we pass to a subtopos $\mathscr{E}$, the condition that $()^{1/T}$ take sheaves to sheaves may be a drastic restriction on a representable T, requiring indeed that T be qualitatively tinier. For example, the function space $(T^T)^X$ on $X = T$ is often finite-dimensional (but see [4]), whereas on other representables, such as $X = T^T$, the function space is typically infinite-dimensional. It is indeed only at the latter level that the examples of real algebraic, real analytic, C^∞ toposes begin to differ, since the maps between the ATOs T^n themselves are the same algebraic ones in all such examples. The term "atom" was used by Bunge for a related notion and later by Kock in his book [8], although it clashes with the use of that term by Barr which applies mainly in a Boolean context. The resulting acronym ATO will be completed below by attributing a physical context, matter and motion, to the M.

axiomatic presentation of these many additional strong properties of T still needs to be worked out; these additional properties are only occasionally needed in this paper.

In the basic setting a great number of important concepts can be defined, such as open and closed inclusion, etale map, micro linear object, manifolds, formal function, strong Nullstellensatz, etc., and especially the foundationally important category $\mathcal{S}$ of T-discrete objects that permits analyzing and constructing general spaces ("Mengen") in terms of natural "structure" arising on discrete objects ("Kardinalzahlen"), as in the Dedekind–Cantor–Hausdorff tradition. Here the natural definition of "discreteness" is $S \in \mathcal{S}$ iff $S = S^T$ canonically; this means that S does not have enough cohesion to permit any non-constant paths, even infinitesimal paths. One partial expression of the idea that T "infinitesimally generates" $\mathcal{E}$ would be the condition that the identity is the only Grothendieck modal operator j on the truth-value space Ω such that T is a j-sheaf.

There are many examples of such settings, for example those provided by C^∞ spaces, real analytic spaces, algebraic spaces, and as yet unexplored possibilities such as an "algebraic" approximation to C^∞ behavior which would be based on adjoining standard "bump functions" like $\exp(-1/\ )$ to the usual algebraic theory of real polynomials. Properties that distinguish these examples can be expressed in the language of the setting $\mathcal{E}$, T but our general considerations here will not depend on these properties. In all those examples, T has been defined to be a specified part of a rig R deemed to parameterize a smooth line; conversely, in our general setting we can define

$$D_n(T) = \{h | h^{n+1} = 0\} \subseteq T^T$$

as parts of the canonical bipointed monoid and consider that a *unit of time* is any given isomorphism $T \xrightarrow{\sim} D_1(T)$. Note in particular that in some well-adapted examples T^T may contain objects which are much larger than these D_n, yet still ATOs, for example the germ representor [1].

The fractional exponents may be considered as left actions by "distributional objects" where by the latter we mean any left $\mathcal{S}$-adjoint endofunctor $\mathcal{E} \to \mathcal{E}$. Such may be considered as "random endomorphisms" of $\mathcal{E}$ in the sense that they assign to every point of $\mathcal{E}$ a distribution on $\mathcal{E}$. Ordinary objects A covariantly determine distributional objects via

$$AX = A \times X$$

but any ATO T also contravariantly determines a distributional object $1/T$:

$$(1/T)X = X^T.$$

Composition of objects AB is commutative, since it is given by cartesian product, but composition of general distributional objects is not commutative. For example, if we define

$$(A/T)X = A(1/T)X = A \times X^T$$

then $(1/T)A = A^T/T$.

For any ATO T, $1/T$ is a special distributional object, indeed the inverse image of an (essential) geometric endomorphism of the topos $\mathcal{E}$.

The distributional objects generalize both endomorphisms of $\mathscr{E}$ and sheaves ("$\mathscr{S}$-valued functions") A on $\mathscr{E}$ just as linear operators on a function space generalize both endomorphisms of and functions on the domain space. The category of distributional objects has pushouts, extending the pushouts of actual objects, since the pullback of the corresponding right adjoints serves as right adjoint. A distributional object is called "pointed" if it is equipped with an $\mathscr{S}$-natural transformation from the identity; again, this extends the notion of pointed object. For example, $1/T$ is pointed for any ATO T whether T itself is pointed or not.

In general, a retraction $E^B \to E$ for the diagonal may be called an "averaging process over B", so that for an ATO B, a process of averaging over B is the same as an action by the pointed distributional object $1/B$.

Countable coproducts of distributional objects should exist too, hence also free actions extending the notion of free monoid as the countable coproduct of iterates; however, we will only need to assume this for some very special cases involving distributional objects G obtained by "collapsing to a point" a copointed part P of a fractional object $F = A/T$

$$\begin{array}{ccc} P & \longrightarrow & F \\ \downarrow & & \downarrow \\ 1_{\mathscr{E}} & \longrightarrow & G \end{array}$$

i.e. for every X

$$\begin{array}{ccc} PX & \longrightarrow & A \times X^T \\ \downarrow & & \downarrow \\ X & \longrightarrow & GX \end{array}$$

is a pushout in $\mathscr{E}$. For a pointed distributional object G, a further colimit yields the distributional monoid whose actions are those of G respecting the pointing. In fact, a basic example of a co-pointed distributional object is $P = T/T$ with the evaluation map $T \times X^T \to X$ as the co-pointing. To make this P a "part" of F we use the additional data of a map $T \xrightarrow{\alpha} A$, for then there is the induced $(T/T)X = T \times X^T \to A \times X^T = (A/T)X$ given by $\alpha \times 1$. The pushout G_α in this case represents an important notion:

Proposition 1. *For any map $A_\text{o} \xrightarrow{\alpha} A$, by an α-prolongation operator or α-law on X is meant any section $X^{A_\text{o}} \xrightarrow{\xi} X^A$ for α^*. If A_o is an ATO and M_α is the distributional monoid generated by the pointed distributional object G_α constructed as above, then M_α actions are the same as α-laws and the category of all these is a topos $\mathscr{E}^\alpha$ with a "surjective" morphism*

$$\mathscr{E} \to \mathscr{E}^\alpha$$

which is $\mathscr{S}$-essential.

Proof. By the Eilenberg–Moore proposition, $\mathscr{E}^{\alpha}$ is also the category of co-actions of the comonad (involving fractions as exponents) right adjoint to M_{α}, so in particular $\mathscr{E}^{\alpha}$ is lex comonadic over $\mathscr{E}$ (with the inverse image functor being the one neglecting the prolongation operator that each object of $\mathscr{E}^{\alpha}$ has). (An alternate proof and several interesting further developments are given in the article [9] by Kock and Reyes.) □

Note that the above proposition holds for any $A_o \xrightarrow{\alpha} A$ where A_o is any ATO (A_o is not necessarily the basic ATO T of our setting). It also holds for any object A, for example $A = 1$; in that case the category $\mathscr{E}^{\alpha} = \mathscr{S}$ and the extra left-adjoint in the essential surjection is a "connected components" functor which actually preserves finite products. (Note that S^X is T-discrete for any T-discrete S and for any X.)

Since the product of two ATOs is an ATO, it follows that for any ATO I the product $\alpha \times I$ is also a datum such that the prolongation operators along it form a topos.

Proposition 2. *There is a functor $\mathscr{E}^{\alpha\times I} \to \mathscr{E}^{\alpha}$ assigning, to any $(\alpha \times I)$-law on X, an α-law on X^I. The natural structure of the composite functor $\mathscr{E}^{\alpha\times I} \to \mathscr{E}^{\alpha} \to \mathscr{E}$ includes action by the monoid I^I.*

In case we have two data $B_0 \xrightarrow{\beta} B$ and $C_0 \xrightarrow{\gamma} C$ we can inquire about functors $\mathscr{E}^{\gamma} \to \mathscr{E}^{\beta}$ that preserve the underlying space. Such are induced by morphisms of distributional monoids $M_{\beta} \to M_{\gamma}$, that might in particular be presentable by morphisms $G_{\beta} \to G_{\gamma}$ of pointed "objects"; how can morphisms of pointed objects of that special kind be induced? A morphism $\sigma : \beta \to \gamma$ of morphisms is not sufficient, because of the mixed variance, nor could a "twisted" morphism alone be guaranteed to preserve the fundamental prolongation identity. Sufficient will be the following.

Data. Consider a triple of maps σ_0, σ_1, and σ_{-1} such that σ_0 and σ_1 determine a morphism from β to γ

$$\text{i.e. } \sigma_1\beta = \gamma\sigma_0 \qquad \begin{array}{ccc} B_0 & \xrightarrow{\beta} & B \\ \downarrow{\scriptstyle \sigma_0} & & \downarrow{\scriptstyle \sigma_1} \\ C_0 & \xrightarrow{\gamma} & C \end{array}$$

and such that σ_{-1} is a retraction for σ_0: i.e. $\sigma_{-1}\sigma_0 = 1_{B_0}$.

Proposition 3. *These data induce a configuration-space-preserving functor $\mathscr{E}^{\gamma} \to \mathscr{E}^{\beta}$ between the categories of prolongation operators. This is merely a remark about the algebra of categories and does not depend on B_0, C_0 being ATOs. In case B_0, C_0 are ATOs, then σ induces a map of presentations*

$$\begin{array}{ccccc} B/B_0 & \longrightarrow & G_{\beta} & \longrightarrow & M_{\beta} \\ \downarrow & & \downarrow & & \downarrow \\ C/C_0 & \longrightarrow & G_{\gamma} & \longrightarrow & M_{\gamma} \end{array}$$

Proof. Actually $X^{(\,)}$ could be any contravariant functor, denoted *. We must show that if ξ is any section for γ^*, $\gamma^*\xi = 1$, then

$$\xi^{\#} =_{\text{def}} \sigma_1^*\xi{\sigma_{-1}}^*$$

is a section for β^*. But

$$\beta^*\xi^{\#} = \beta^*\sigma_1^*\xi{\sigma_{-1}}^* = \sigma_0^*\gamma^*\xi{\sigma_{-1}}^*$$
$$= \sigma_0^*{\sigma_{-1}}^* = (\sigma_{-1}\sigma_0)^*.$$

The map of presentations is

$$\begin{array}{c} B \times X^{B_0} \\ \Big\downarrow {\scriptstyle \sigma_1 \times {\sigma_{-1}}^*} \\ C \times X^{C_0} \end{array}$$

compatible with β and γ because of the role of σ_0. We will show in the theorem below that any such induced functor has a left adjoint. □

A special case of the above has $B_0 = C_0$ and $\sigma_{-1} = 1 = \sigma_0$; the only condition is $\gamma = \sigma\beta$. Still more special is that where $B_0 = A_0 \times I$, $B = A \times I$, $\beta = \alpha \times I$ and $A \times I \xrightarrow{\sigma} C$ is such that $\gamma = \sigma(\alpha \times I)$. Then the category of γ-laws maps into the category of α-laws, changing any underlying space X of "configurations" into the special space X^I of "states". The contrast between these two terms arises because of this change of categories. For example, in such a context $A_0 \times I \to A \times I \xrightarrow{\sigma} C$ often arises when σ is addition of nilpotents as in Examples 3, 4, and 6 below.

Definition. Suppose $A_1 \xrightarrow{\alpha} A_2$ is a given map in $\mathscr{E}$. An operator $X^{A_1} \xrightarrow{\xi} X^{A_2}$ is a *prolongation law of order* α on the *configuration space* X iff $\alpha^*\xi = 1$, the identity on X^{A_1}. A *lawful motion* is a morphism in the category $\mathscr{E}^\alpha$ of α-laws, that is a map $X \xrightarrow{f} Y$ between configuration spaces with $f\xi_X = \xi_Y f$. If L is a monoid acting on both A_1 and A_2 and commuting with α, then a law ξ is an *L-connection* iff $\lambda_2\xi = \xi\lambda_1$ for all λ in L, where λ_k denotes the action of λ on X^{A_k}.

Example 1. If $A_1 = 1$, $A_2 = T$, and the unique α $1 \xrightarrow{o} T$ is the basic point-contradiction, then a prolongation law of that order is a first-order ODE, that is just any action $T \times X \to X$ of the pointed object, also known as a vector field on X. Since $\mathscr{E}$ has free monoid objects generated by pointed objects, and hence free and cofree actions, these laws and lawful motions form a topos $\mathscr{E}^T$ with an $\mathscr{E}$-essential geometric morphism $\mathscr{E} \to \mathscr{E}^T$ whose inverse image is the functor that forgets the prolongation law (or vector field) and remembers the underlying configuration space. In this example configuration space and "state space" will mean the same thing. The monoid T_∞ generated by the pointed object T naturally has an action of the multiplicative monoid R of point-preserving endomorphisms of T. Hence there is a semidirect product (fibered category) monoid

$$T_\infty \hookrightarrow R \ltimes T_\infty \to R$$

whose actions may be of interest.

Example 2. All of Example 1, except for the terminology, applies to *any* pointed object; in particular, $\mathscr{E}^V$ is an $\mathscr{E}$-topos and $\mathscr{E} \to \mathscr{E}^V$ is $\mathscr{E}$-essential without any condition on the pointed object V. This contrasts with the more general examples below, which typically involve only $\mathscr{S}$-toposes and $\mathscr{S}$-essentiality.

Example 3. If D_k is a space of kth order nilpotent infinitesimals, and if α is the inclusion $D_{n-1} \hookrightarrow D_n$, then a prolongation law of order α is what is usually called an nth order ODE.

Example 4. If $A_1 = T$ and if A_2 is in some sense second-order, with α an inclusion, then an α-prolongation law may be called a second-order ODE. There are several precise choices for A_2:

- the symmetric power $T^2/2!$ with order $\alpha =$ the quotient map preceded by either coordinate axis,
- the coequalizer of the coordinate axes $0 \times T$ and $T \times 0$ (this has only one meaning because T has only one point)

  $$T \rightrightarrows T \times T \longrightarrow C$$

  with order $\gamma =$ the common composite,
- a generic quantity h with $h^3 = 0$, i.e. the object $D_2 \subset T^T$; here the specification of order requires a unit of time and addition in R.

There is clearly a natural map $C \to T^2/2!$ taking γ to α because the group 2! interchanges the two coordinate axes; in most examples this map can be assumed invertible, since it is stable under base change, intuitively a further aspect of the "amazing tinyness" of T. When the pointed monoid $R \subset T^T$ happens to have a uniquely determined homogeneous addition, a choice of unit of time induces a map $T^2/2! \xrightarrow{s} D_2$, so that second-order ODEs of order $s\alpha$ give rise to ODEs of order γ, in other words, in that case second-order is interpretable as iterated first-order. Note that "order" is here more precise than a mere natural number. Caution: the proper interpretation of the homogeneous map s requires that the element $2 \in T^T$ be invertible. In many examples s is also an isomorphism. Note that the monoid $L = R$ naturally acts on all three $C \to T^2/2! \to D_2$, so that in any one of the three toposes there is a subcategory of L-connections, usually known as *affine connections*. This subcategory is monadic by a quotient monad.

Remark. If we use explicitly the assumption that T has the property that its tangent vectors can be added, then comparisons

$$T^n/n! \to T^T$$

mapping into $D_n(T)$ can be defined. But conversely, much of the small role addition plays in our considerations here can be played by the maps into the symmetric powers themselves. In other words, without assuming any addition in the tangent fiber R of T, we can instead interpret iteration of infinitesimal time translations as formal addition in the commutative monoid freely generated by the pointed object T, which strikingly is the same as the "non-commutative" monoid generated by the same data.

Example 5. Prolongation laws are not the same as "force laws"; in typical examples, laws form merely an affine space whereas force laws form a vector space, indeed the vector space of translations of that affine space. Forces are a measure of the contrast between the actual law and a second auxiliary law often considered to express inertia. For ways of constructing a possible auxiliary law from the actual law with help of a mass distribution see part II. This leads to a very important further interpretation τ of the idea of "second-order", namely the pushout

$$\begin{array}{ccc} & T' & \\ \nearrow & & \searrow \\ T & \xrightarrow{\quad\tau\quad} & T'' \\ \searrow & & \nearrow \\ & T' & \end{array}$$

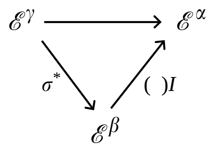
of two copies of a given notion of second order (the resulting T'' actually contains some third order nilpotents in the standard example). Clearly, an object in the category $\mathscr{E}^{\tau}$ of prolongation laws of order τ is a single configuration space X equipped with two laws of order $T \to T'$. Such we may reasonably call a *Galileo–Newton dynamical law*, a name justified with respect to the aspect of pure dynamics; for the material aspect see part II. Often one requires that the auxiliary law be an affine connection, i.e. R-homogeneous, thus distinguishing it from the main law.

Remark. For any Galileo–Newton dynamical law, *force* can be defined as any measurement of the difference between its two component laws. In the frequent case where all the possible laws form an affine space, there is a natural largest ideal force measurement, namely subtraction going into the corresponding vector space of translations. In general, a force measurement is only partial given by an apparatus consisting of a map over X, $X^{T_2} \xrightarrow{\phi} F$, considered as a *force component*; then $\phi\xi$ is the force in direction ϕ that accompanies the Galileo–Newton dynamical law ξ. If F has addition and subtraction even though X^T does not and if ϕ vanishes where the two laws agree, then this would permit the discussion of approximating laws by (virtual) affine combinations of simpler laws, as is fundamental for engineering [6].

While it is reasonable to speak of arbitrary morphisms in a category $\mathscr{E}^{\alpha}$ as lawful "motions", it is customary to reserve that term for morphisms whose domain is a special object considered as an interval of time U structured itself with its flow, a given α-law $A \times U^{A_0} \to U$. For example, if $A_0 = 1$, $A = T$, we might take U to be an open one-dimensional interval of time, or the still smaller D_∞ that represents "formal" solutions and is closed, like U, with respect to infinitesimal translation $T \times D_\infty \xrightarrow{+} D_\infty$. We are interested, however, not only in general vector fields, but also in the more highly structured ODE's, especially in the crucial case of second-order laws.

Theorem (The algebra of time).

Suppose σ_{-1}, σ_0, σ_1 are data that present a homomorphism $M_\beta \to M_\gamma$ of generalized monoids, where $A_0 \xrightarrow{\alpha} A$ and $C_0 \xrightarrow{\gamma} C$ with A_0, C_0 ATOs, *and I any* ATO, *and where $\beta = \alpha \times I$. Then the composite functor*



has a left adjoint. Hence, given an object U in $\mathscr{E}^\alpha$ whose underlying space is deemed to represent a notion of path or process, one obtains, by applying the left adjoint, a space $\sigma_!(I \cdot U)$ with a γ-law representing the functor "space of solution curves for γ-dynamical systems"

$$\mathscr{E}^\gamma \longrightarrow \mathscr{E}^\alpha \xrightarrow{(U,-)} \mathscr{E}.$$

Note that a category $\mathscr{E}^\alpha$ of prolongation laws is enriched in $\mathscr{E}$, not only in $\mathscr{S}$, since it is an equational class of algebras each involving an A-tuple of A_0-ary operations; this is true for any A_0, not necessarily an ATO, even if $\mathscr{E}^\alpha$ is not a topos.

Proof. As the notation suggests, we can construct the adjoint as a composite of two adjoints $\sigma_!$ and $I\cdot$ for σ^* and $()^I$, respectively. □

An important example of the above constructions involves just a given map.

Definition. Given $C_o \xrightarrow{\gamma} C$, let $T(\gamma)$ be C^{C_o} considered as a pointed object pointed by $[\gamma]$. If $C_o = 1$, then $T(\gamma)$ remains γ, for example if $C = T$, the basic ATO. Call this pointed space $T(\gamma)$ the *microtime*.

Corollary. *Given any γ-law ξ on a configuration space X, there is an associated action by the micro time $T(\gamma)$ on the space X^{C_o}. This action*

$$\xi_\tau[x](h) = \xi[x](\tau(h))$$

is special in that it satisfies the equations

$$\tau_1(h_1) = \tau_2(h_2) \Rightarrow \xi\tau_1(x)(h_1) = \xi\tau_2(x)(h_2)$$

for micro times τ and for h in C_0.

Example 6. In case $C_o = D_{n-1}$, $C = D_n$, $\gamma =$ the inclusion, and $h_i \varepsilon D_i$ where $i_1 + i_2 \leqslant n$, then we can take for example τ_1 to be translation by h_2 and τ_2 to be translation by h_1 to recover the well-known equation which characterizes higher-order laws as special first-order laws on a state space like X^{C_o}. However, there is more structure in micro time beyond this possibility of acting by translation: Via the given map γ, the monoid $C_0^{C_o}$ is included into the pointed space $T(\gamma)$, so that the $T(\gamma)$-action induces a $C_0^{C_o}$-action. Those actions arising from γ-laws thus actually permit recovery of the configuration space from the state space. Our central example has $C_o = T$ and $C = T'$, second order, so that $T(\gamma)$ is no longer infinitesimal, but rather we have

$$T \times R \subset T(\gamma) \subset T' \times R.$$

Nonetheless, the $T(\gamma)$-actions (with suitable relations—see Proposition 6) will form even an $\mathscr{E}$-topos.

Proposition 4. *The fixed-point space of the codiscrete action of a monoid on a space* X^{C_o}, *where* C_o *is the nonempty space of constants of the monoid, is* X *itself. For example, for the full endomorphism monoid of any given space* C_o, *the constants are just* C_o.

For a given law one often seeks *solutions* to the initial and boundary value problems it poses. These solutions are again prolongation operators of a different sort conditioned by the law. If U,X are objects in $\mathscr{E}^\gamma$ (i.e. configuration spaces equipped with γ-laws), and if B in $\mathscr{E}$ is a subspace of the underlying space of U, there is the restriction map

$$\mathscr{E}^\gamma(U,X) \to \mathscr{E}(B,X) = X^B$$

and a *solution* to the B-problem is just a section for that restriction map. Obviously, if B_1 and B_2 are such that there is a γ-connection, that is, an isomorphism

$$\mathscr{E}(B_1,X) \xrightarrow{\sim} \mathscr{E}(B_2,X)$$

induced by a solution to the B_1-problem, a solution to the B_2 problem results. The process of applying a B_1 solution via a connection in order to solve the B_2 problem is called *shooting* in the special case where B_1 is an ATO but B_2 is discrete. If the configuration space X is affine and has a connection as given by the Kock–Lawvere axiom, then the shooting is straight. The term "shooting" apparently has the following origin: In order to be free to shoot an arrow from any given place with assurance of hitting any given target (these two conditions being given by a map $B_2 \to X$), I must recognize the physical necessity expressed by the specific γ-law that applies and on that basis choose the appropriate initial condition (e.g. velocity vector) $B_1 \to X$.

For any topos such as $\mathscr{E}^\gamma$ and object Y in it, the "slice" topos $\mathscr{E}^\gamma/Y$ is again a topos. For example, non-autonomous systems over a time interval U can be described this way by taking $Y = F_\gamma(U)$, the relatively free γ-dynamics generated by U. The latter is merely equipped with the translation vector field $T \times U \to U$. Here we assume that γ is equipped with a morphism from the first order $1 \to T$. Essentially we model the configuration space E to include a clock, and the equivariant structure morphism $E \to Y$ reads the clock. Bigger parameter spaces Y arise when the system to be described involves the influence of an environment such as body forces, radiation, forcing terms, frequency parameters etc. "anything you want", as Lang wrote about slice categories in his 1960 review of Grothendieck's EGA project, provided that these are also subject to specified laws.[2]

Models for things moving under the influence of such an environment Y constitute $\mathscr{E}^{\gamma}/Y$. For example, locally the configuration space may split as a product $E = X \times Y$ where X^T is the space of internal states; relative to some affine inertia law, the force–density law corresponding to a γ-prolongation law can then be written in the traditional way

$$\ddot{x} = f(x, \dot{x}, y, \dot{y})$$

$$\ddot{y} = g(y, \dot{y}).$$

The crucial condition for the projection $X \times Y \to Y$ to be γ-lawful is that g does not depend on X^T. For example, if y is mere one-dimensional time, it would be reasonable to take $g \equiv 0$. But if y is the altitude of some falling device whose internal motions may be effected by y, even though y does not vary too much, we might take $g = \text{const}$, etc. Although such an asymmetric model of the relation between a thing and its environment is an appropriate approximation in many circumstances, it cannot be of the canonical Hamiltonian form, as the following shows:

Proposition 5. *If a product space $X \times Y$ carries a dynamical law given by a Hamiltonian function H, then if one of the projections $X \times Y \to Y$ is equivariant, so is the other $X \times Y \to X$. "No reaction implies no action".*

Proof. We are supposing that locally X^T and Y^T trivialize as $X \times V$ and $Y \times W$, respectively. Then

$$\dot{x} = \partial_v H, \qquad \dot{y} = \partial_w H,$$

$$\dot{v} = -\partial_x H, \qquad \dot{w} = -\partial_y H$$

and the assumption requires that $\partial_w H$ and $\partial_y H$ be independent of x, v, so that

$$\partial_x \partial_w H = 0, \qquad \partial_x \partial_y H = 0,$$

$$\partial_v \partial_w H = 0, \qquad \partial_v \partial_y H = 0.$$

Since partial derivatives commute, these equations imply also that $\partial_v H$ and $\partial_x H$ are independent of y, w, so that $X \times Y \to X$ is also equivariant. □

[2] In this connection, note that in Indo-European languages "time" is not typically just an abstract one-dimensional continuum; the abstract time is a dialectical negation of the idea of time as the rich environment of external conditions that may influence our system, but which we can influence only negligibly: In Italian *tempo* (time) also means weather, in Danish the word for time is *tid*, which is old English for tide. In a *Zeit-ung* like the *Times* the tidings describe the whole tempestuous march of events over which the reader has little control. In English we have "the worst of times" and "the best of times", and "the times are a-changing", something a mere smooth line cannot be or do.

It follows that the energy is a sum function $H = H_X + H_Y$, describing the product of two Hamiltonian systems in the category $\mathscr{E}^\gamma$ of all second order systems, Hamiltonian or not.

II

The notion of *averaging* is dual to that of prolongation in the sense of the following.

Definition. Given a map $B \xrightarrow{\alpha} A$, an *α-averaging functional* on E is a map

$$E^B \xrightarrow{m} E^A$$

for which $m\alpha^* = \text{identity}$, i.e. a retraction onto E^A for the inclusion α^*. Of course there will not exist any averaging functionals if α^* is not an inclusion.

Remark. An example involves the space E_0 in which the solar system moves, with B the sum of the main extended bodies (sun, planets, and significant moons) and α the partitioning of B into a finite discrete set A of point-idealizations. The barycentric calculus was developed by an astronomer (1827). It has become traditional to consider averaging functionals m that commute with another given averaging process ℓ on another "body" like 2 or 3; such averaging processes m are called ℓ-linear.

Proposition 6. *An averaging functional on X is equivalent to an action on $Y = X^B$ of the monoid presented by*

$$(B^B * \{1, e\})/e\lambda e = \lambda e \text{ for } \lambda \text{ any endomorphism of } \alpha.$$

The dual notion to "L-affine connection" is that of an averaging functional satisfying also $e\lambda e = e\lambda$ for λ in L. Dually, for $C_0 \xrightarrow{\gamma} C$, a γ-prolongation on X is equivalent to an action on X^C of a monoid $C^C\{e\}$, where the adjoined idempotent is subjected to opposite defining equations, so that in particular

$$(e\lambda)(e\mu) = e(\lambda\mu).$$

The distribution of the mass of a body B induces an averaging operation along $B \to 1$ on a space E where B is moving

$$E^B \xrightarrow{m} E.$$

Thus we are led to consider configuration spaces X (like E^B) that have an idempotent operator embodying such a center-of-mass averaging. Given a γ-dynamics on X, we deduce a γ-dynamics ξ_o on the fixed space E of that action by (roughly) $\xi_o = m\xi i$, where $mi = 1$ is the splitting of the idempotent. Then there is an induced dynamics $\xi_o B$ on any power E^B. A possible axiom on the relation of γ and m is that the dynamics ξ_o on E (and hence ξ_o^B) is actually L-homogeneous; this may have been what Hertz intended.

Proposition 7. *Given a γ-prolongation law $X^T \xrightarrow{\xi} X^{T'}$ and a retraction $X \xrightarrow{m} E$ of the configuration space onto the subspace $E \xrightarrow{i} X$, the map $E^T \xrightarrow{\xi_0} E^{T'}$ defined by*

$$\xi_0 = m^{T'} \xi i^T$$

is a section for γ^* on E, i.e. a γ-prolongation law on E. However, neither m nor i is an equivariant map (lawful motion) between ξ and ξ_0. In case $X = E^B$ with i the diagonal, m and ξ induce ξ_0^B, a second structure on E^B so that together ξ and ξ_0^B determine a γ-prolongation law relative to the self-pushout of $T \xrightarrow{\gamma} T'$ as in Example 5.

Remark. For a Galileo–Newton system that arises from a mass distribution this way, the auxiliary law deserves to be called inertial, so that force, mass, and acceleration have their traditional sort of relationship.

Remark. Since prolongation laws and averaging functionals both involve splitting (in opposite senses) of natural maps, yet themselves are usually not natural, the idempotents they give rise to are in a sense half-natural. That can be expressed algebraically in terms of semi-commutation rules. For example, suppose we are given both a mass distribution and a law of motion in the following relation:

Example 7. Suppose X is the configuration space for a $C_0 \xrightarrow{\gamma} C$ prolongation law ξ, but that $X \underset{i}{\overset{m}{\rightleftarrows}} E$ is a retraction. Then X^C (the space of slightly longer paths) carries a left action by the six-element monoid generated by two idempotents $a = \xi\gamma^*$ and $\mu = i_* m_*$ satisfying the relation

$$a\mu a = a\mu.$$

Thus $v = a\mu$ is again idempotent, but $\psi = \mu a$ satisfies only $\psi^3 = \psi^2$ in general. It is reasonable to consider the other m_*

$$X^{C_0} \to E^{C_0}$$

as the passage from fine states to coarse states (or micro states to macro states), keeping in mind that m_* is not an equivariant (lawful) map from ξ to ξ_0. It may be possible to choose (following Muncaster [10]) a section i (possibly not "diagonal") that is equivariant. That condition is equivalent to the further relation

$$\mu\xi\mu_0 = \xi\mu_0,$$

where μ_0 is the obvious idempotent on the space of fine states. Just as the action of $T(\gamma)$ is a significant portion of the algebraic structure of the state space functor, so the action of this monoid is part of the natural algebraic structure of the micro-path functor $()^C$.[3]

In order to begin to illustrate how the categorical algebra discussed in this paper can help to provide a smooth objective background for dealing with particular kinds of constitutive relation for materials, consider the general idea that an ATOM is an ATO moving, i.e. a body with a mass distribution and a dynamics as above, where the body is itself an ATO. One possible model making this idea precise is the following

Definition. Let A be an amazingly tiny object and let $T \xrightarrow{\tau} T''$ be a given "order" with T amazingly tiny. Then a generalized *A-monatomic system* is a space E equipped with an averaging process $E^A \xrightarrow{m} E$ and a τ-prolongation law ξ on E^A as configuration space.

Example 8. If we have a family S of identical atoms A and if E_0 is "ordinary space", then a natural configuration space for the body $S \times A$ is

$$X = E_0^{S\times A} = E^A,$$

where $E = E_0^S$. If an isolated single atom has mass distribution $E_0^A \xrightarrow{m_0} E_0$ for its placements in ordinary space, then a suitable averaging process along the projection $S\times A \to S$ is induced by $()^S$ as

$$E^A = E_0^{S\times A} \cong (E_0^A)^S \xrightarrow{m_0^S} E_0^S = E.$$

Of course, if we also had a distribution on S, leading to another averaging $E \to E_0$, we could define a center-of-mass notion for the whole body $B = S \times A$ and its placements in ordinary space; but that coarser, kinetic theory sort of functional is not needed for our construction here. Rather, the much more refined $m \cong m_0^S$ essentially just averages over the internal configurations of each atom, but remembers the precise mutual configuration of the resulting virtual point-particles. The "generalization" in the above definition means that we do not assume that m arises from some single smaller m_0 in this way, nor even that E is a function space E_0^S. The state space for a general

[3] In general, one might say that an important purpose of categorical algebra is to express everything possible in terms of adjointness, naturality (homomorphicity) etc., in order to provide a smooth context for studying more serious contradictions; then cohomology measures the extent to which topos morphisms are not local, essentiality measures the extent to which spaces are not connected, and adjoints between $\mathscr{E}$-enriched categories may be merely $\mathscr{S}$-enriched; there is much information in those gaps. An example of the importance of such non-naturality is well-known in statistics and mechanics under the name of "moments". Namely, suppose $E^S \xrightarrow{m} E$ is a retraction for the diagonal. If $E^n \xrightarrow{\theta} E$ is any map, one can ask whether θ commutes with m. In case θ is also a retraction for the diagonal into E^n, one could then say that m is θ-linear; often m is θ-linear for a reasonable set of averaging processes θ which are linear among themselves. But if $n = 1$ and θ is "quadratic", then m commutes with θ only if all possible data has m-variance $= 0$; that tends to imply that m is concentrated to evaluation at a point of S, but then (by naturality) m will commute with all operations θ. However, more typically, (as an instance of semi-homomorphicity) some particular data in E^S may have zero variance although most do not. Similarly, for $n = 2$ and for θ a "multiplication" on E, the equation $m\theta_S = \theta(m \times m)$ usually does not hold, but the two sides may give equal values on some special "θ-uncorrelated" pair $\langle x, y\rangle$ in $E^S \times E^S$. Thus a statistic $E \times E \xrightarrow{\varphi} V$ vanishing on the diagonal, etc. (as in the concept of force component) yields a measurement $\varphi\langle m\theta_S, \theta(m \times m)\rangle$ of θ-correlation.

A-monatomic system is of the form $X^T = (E^A)^T = (E^T)^A$, where E^T is the tangent bundle of E.

Theorem. *The A-monatomic systems of order τ in a base topos $\mathscr{E}$ form themselves a topos, defined over the base topos $\mathscr{S}$ of spaces which are both T-discrete and A-discrete, and receiving a "surjective" $\mathscr{S}$-essential geometric morphism from $\mathscr{E}$.*

Proof. As already remarked, an internal mass distribution of the A amounts to an action of the pointed distributional object $1/A$. Since both T and A are ATOs, $T \times A$ is as well, so that prolongation laws of order

$$T \times A \xrightarrow{\tau \times 1_A} T'' \times A$$

are an example of the kind already treated in part I; but since such prolongation laws on E are exactly the same as τ-laws on E^A, we can form the pushout of orders, of pointed distributional objects, and of monads, etc., to find the adjoint monad whose algebras form the topos required. □

III

Freyd proved that for any ATO D in a topos $\mathscr{E}$, the object $D \times B$ is an ATO in the topos $\mathscr{E}/B$, though usually not an ATO in the sense of $\mathscr{E}$ itself. Nonetheless, the object $D \times B$ is important in $\mathscr{E}$. For example, if S is two-dimensional, then $S' = T \times S$ has three-dimensional tangent space; if we consider S' as a body that can be configured via placements in three-dimensional space E_0, we may try to find laws ξ of motion on $X = E_0^{S'}$ appropriate to the sort of body known as a classical Cosserat continuum: a body so extremely thin that it responds always as a flexible sheet; it can experience stresses and strains in all directions like any three-dimensional body. Similarly, a flexible cord may be modeled as $U' = D \times U$ where U is a one-dimensional interval and D is a "two-dimensional infinitesimal disk", an ATO which will serve as the cross-section of U'; the mechanics of such a flexible cord were already studied by Euler. The extreme case where an ATO itself is the underlying space of an infinitesimally three-dimensional "point" was studied by Cohen and Muncaster [3]; some examples are given below. In my opinion attributing "dimension 0" to a discrete sum $\sum A_i$ of ATOs, as is sometimes done in algebraic geometry, is too crude, since more than just the underlying topological space is relevant. The system of dimensions is a more subtle structure than the mere natural numbers and we can attribute a special dimension (ε?) to such sums, unless all the A_i are themselves nearly punctual. That would be a theorem, not just an imposed definition, if the system of dimensions for $\mathscr{E}$ were construed as a universal monoidal poset or universal 2-rig in the spirit of Schanuel [12].

In the theory and practice of the modern science of sophisticated materials, much more general "microstructure" of the Cosserat sort must be considered, perhaps even requiring more additional directors so that the total tangent dimension is greater than three. Capriz in his book [2] lays down on page 10 a general definition of a *continuum with microstructure* as a body B_0, together with another space $\mathscr{M}$ which is equipped with an action by the group G of rigid motions of E_0, but an action which depends only on the rotational part of a motion, and with a special class $\mathscr{C}$ of admissible placements of B_0 into $E_0 \times \mathscr{M}$. Apart from smoothness, it is assumed that $\mathscr{C}$ is closed under the diagonal action of G on $E_0 \times \mathscr{M}$.

While specific constitutive relations, forces, motions, etc. in continuum thermomechanics are mostly expressed in terms of variable extensive and intensive quantities, the character of the underlying domain spaces where those quantities vary is also important. Those spaces have dimension and homotopy type, and their infinitesimal and smooth structure needs to be considered as well. However, all four of those qualities are reduced to nothing if one considers that underlying spaces are merely abstract sets equipped with sigma-algebras. We need to understand when microstructures $\mathscr{M}, \mathscr{C}$ of the sort described by Capriz and by many others can be extracted from a mere object B (whose underlying measurable space may be the same as that of the B_0) in somewhat the same way as in my above account of the Cosserat surface.

Construction. Let B be any object in $\mathscr{E}$, whose tangent bundle is trivialized by $B^T \cong B \times W$ and likewise E_0 with $E_0^T \cong E_0 \times V$. Here W and V are equipped with pointed R-homogeneities to agree with those naturally on the fibers, but they need not have any internal additive structure; however, the inverse trivialization $B \times W \to B^T$ can be denoted by $+$. An endomap β is a *translation* relative to such a trivialization if $\beta^T = \langle \beta, 1 \rangle$.

Definition. $\mathscr{M} = \mathrm{Hom}_R(W, V)$ the homogeneous maps. For any "placement" $B \xrightarrow{x} E_0$ we can take its derivative $x' = \pi x^T$:

$$\begin{array}{ccccc} W \times B & \xrightarrow{x^T} & V \times E_0 & \xrightarrow{\pi} & V \\ \downarrow & & \downarrow & & \\ B & \xrightarrow{x} & E_0 & & \end{array}$$

and then the exponential transpose $\mathrm{grad}(x) = [x']$:

$$B \longrightarrow \mathrm{Hom}_R(W, V) = \mathscr{M}.$$

Then define $\mathscr{C} = \{\langle x, \mathrm{grad}(x) \rangle \mid x \varepsilon \mathscr{C}_0\}$ where $\mathscr{C}_0$ is a specified space of placements of B itself into E_0 itself, closed under G. Since B is richer than a traditional manifold, in this way these placements in $\mathscr{C}_0$ in many cases determine the accompanying microplacements as well; there is no restriction on the dimension of a tangent-space W in B, even if the underlying manifold of B is only 3, 2, 1, or 0-dimensional.

Theorem. *The above construction yields an example of a body with microstructure, if we assume E_0 is ordinary space. In fact, not only G, but all endomaps g of E_0 act on $\mathscr{C}$, provided g preserves $\mathscr{C}_0$.*

Proof. $g \cdot \langle x, a \rangle = \langle gx, \mathrm{grad}(gx) \rangle$ if $a = \mathrm{grad}(x)$. The condition that two endomaps differing only by a translation β act the same on the $\mathscr{M}$-component of $\mathscr{C}$, is verified by the following calculation:

Suppose

$$\bar{g} = \beta g, \quad \beta^T = \langle \beta, 1 \rangle$$

in the given trivialization.

Then

$$\langle \bar{g}, \bar{g}' \rangle = \bar{g}^T = \beta^T g^T$$

$$= \langle \beta, 1 \rangle \langle g, g' \rangle = \langle \beta g, g' \rangle$$

$$= \langle \bar{g}, g' \rangle. \quad \square$$

IV

Although we have referred to "ordinary" differential equations in this discussion, thus far nothing in the formal results requires one-dimensionality of the infinitesimals; our prolongation operators may relate part-icles (qualitatively diminutive parts) of higher-dimensional space. We could impose geometrical conditions on our basic T such as contractibility i.e. T^T connected, which suggests that the dimension of T is not too small, and commutativity of the point-preserving part R of T^T, which suggests that the dimension of T is not too large. Here I heuristically refer to a still not developed theory of even more refined dimensions within the global dimension ε. However, the usual meaning of "partial differential" equations refers to another aspect, namely that the laws themselves have infinitesimal support, with respect to a body over which the configurations are functions or distributions.

The specific examples of such ATOs that may be useful as infinitesimal bodies are typically spectra of both Weil and Gorenstein algebras, that consist of, or are closely associated with, quantities nilpotent of second order ($x^3 = 0$). That is more or less independent of whether the whole topos, or specifically the monoid $(T^T)^{(T^T)}$, consists of algebraic, analytic, or C^∞ maps or something in between.

A question that may be more than a metaphysical speculation is whether the Pythagorean metric on space leads to the second-order modeling of states of becoming as $()^T$ rather than as longer histories or vice versa. In any case, the fundamentally quadratic character of the basic infinitesimal spaces is fairly firmly entrenched, even in non-Euclidean geometry, general relativity, etc. and in the following discussion we just accept it.

Definition. If a space X is equipped with a homogeneity, i.e. an object of the topos of actions of the multiplicative monoid R, then to some maps $X \to Q$, where Q also has a homogeneity, for example $Q = T^T$ or $Q = R$, one can attribute a *degree* in the classical sense:

$$f(\lambda x) = \lambda^d f(x)$$

where $()^d$ is a homomorphism of monoids $R \to R$. The action of $\lambda = 0$ retracts X onto the fixed-point subspace E, and as in the case $E = 1$ we may call that subspace the *origin*. A homogeneous subspace of X is in a *neighborhood of order* $\leqslant n$ of the origin if all R-valued homogeneous functions of degree greater than n vanish when restricted to the subspace.

Proposition 8. *If X is a space with homogeneity and if B is a neighborhood of the origin of order 2, then any function f homogeneous of degree 1 satisfies $f^3 = 0$ on B.*

Proof. Both the multiplication of functions and the acting homogeneity involve at bottom the same composition in T^T, but within the commutative R; thus degrees add in the expected way. □

A fundamental construction of algebraic geometry is the attempt at contravariant reconstruction of a space from the algebra of functions on it, the latter construed as a discrete space equipped with operations. If we consider a space X equipped with homogeneity, with $V_n \subset R^X$ the spaces of homogeneous functions for the various homomorphisms $()^n : R \to R$, in particular $V_0 = R^E$, then on the discrete spaces

$$A_n = \mathscr{E}(1, V_n)$$

we have a symmetric bi-homogeneous "form"

$$A_1 \times A_1 \to A_2.$$

If the whole space X is of order 2, connected ($E=1$) and "affine", then it is determined as the spectrum of the algebra

$$A_0 + A_1 + A_2.$$

Conversely, the category of all quadratic forms can be embedded in commutative(!) algebra this way. The basic case where V_0, V_1, V_2 are all isomorphic to R occurs very often. The ring as such is the cohomology ring of several important spaces, with various gradings. The fact that the gradings of the cohomology rings come from λ-ring structures drastically restricts what the top grade can be, as described by Eckmann in his beautiful paper [5]. Fixing the 0, 1, 2 grading by nilpotency degree, the spectrum of any quadratic "form" can be defined to consist of graded and homogeneous maps into this basic three-dimensional algebra. Two important classes of quadratic "form" can be distinguished.

Definition. A homogeneity space B of order 2 is *Weil* if $V_0 \approx R$ and *Gorenstein* if $V_0 \approx V_2$. The Gorenstein–Weil algebras where $V \approx R^3$ and where the quadratic form has invertible gradient are called *supports of Laplacians* [8].

Proposition 9. *The free Weil space of order 2 on a three-dimensional V has a 10-dimensional function algebra. Its Gorenstein–Weil subspaces of order 2 have five-dimensional function spaces. Smaller subspaces include the first-order Weil algebra with four-dimensional function space and the origin which is a Weil space of order zero.*

The sort of infinitesimal structure described in the proposition, at each point of a continuous body, will be relevant in the analysis of the contact forces involved in a local constitutive response functional. With more involved microstructures, other ATOMs of the kind discussed in part III are active at every point.

One aspect of the unification of continuum physics provided by the smooth topos setting is that to an important extent infinite-dimensional systems such as electromagnetism, finite systems of mass-points, and extended mass-points or particles can be treated in the same way, recognizing of course the differences, e.g. that non-trivial differential operators or laws of infinitesimal support may be relevant. For example

Definition. A scalar mass distribution on a body B is an R-homogeneous retraction $R^B \to R$ onto the constants.

Here the retraction condition means that we have normalized the total mass to be 1, appropriate if we are not considering other bodies, and we are of course using the classical Riesz interpretation of extensive quantities as integration processes on intensive quantities. Also, positivity requirements not discussed here would be required to distinguish mass from charge.

Proposition 10. *Even on a one-point body, scalar mass distribution is not a scalar.*

Proof. That is, the mass distribution is not just evaluation at the point, nor is it determined by its total (= the integral of the function 1). Specifically, if R^B is the 10-dimensional Weil algebra referred to in Proposition 9, then evaluating the mass distribution just at the quadratic top-degree portion yields an element of the six-dimensional space of possible Euler tensors [3]. □

Remark. The dimension of the state space $(E_0^B)^T$ of the typical quadratic ATOMs B mentioned is 60, or 30 if one models B merely as a Gorenstein Laplacian support. That of course assumes the simplest tangent bundle model of states of becoming; longer histories would involve more states. The constitutive response functionals for a general body include, as an ingredient of their domain, the mutual forces between the parts of the body. Noll [11] considers those interactions to be described by a vector-valued bi-measure on the body; as before, let us interpret such extensive quantities themselves as functionals on intensive ones, and also interpret a bi-measure as an extensive quantity on the product $B \times B$. Assuming that the vectors are three-dimensional, we see that the dimensionality of the space

$$\mathrm{Hom}_R(R^{B\times B}, V)$$

of such internal interactions of our ATOM B is 300, or

$$3 \times \frac{10^2 - 10}{2} = 135$$

if we consider only anti-symmetric (action = reaction) interactions. Although it has been customary since von Neumann to consider that the state space of even a single particle is infinite-dimensional, these numbers show that at least for engineering purposes a vast variety of constitutive relations is available at the quadratic ATOM level.

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