For Perugia Studies in Mathematics (Proceedings of the May 1997 Meeting in Perugia)
(October 1998 correction)

Volterra's functionals and covariant cohesion of space

F. William Lawvere

Abstract:

Volterra's principle of passage from finiteness to infinity is far less limited than a linearized construal of it might suggest; I outline in Section III a nonlinear version of the principle with the help of category theory. As necessary background I review in Section II some of the mathematical developments of the period 1887-1913 in order to clarify some more recent advances and controversies which I discuss in Section I.

I

The immediate impetus to this historical exploration came from two articles by Gaetano Fichera. The resulting line of study needs to be deepened and considered in more detail, but it already supports certain conclusions concerning the precise methodological direction of global analysis. These methodological conclusions can perhaps be tested on the ample material provided by the important new book by Kriegl and Michor, just published by the AMS [17]. On that basis we can hopefully begin a serious reply to the challenge of 'I difficili rapporti fra l'analisi funzionale e la fisica matematica'. Some of those difficulties are outlined in the article [8] (with that title) by Fichera.

The other article by Fichera, 'Vito Volterra and the birth of functional analysis' [9], is basically a re-affirmation of the role of the Volterra school in the development of modern analysis, in response to Dieudonné's [6] :

> We must finally mention the first attempt at 'Functional Analysis' of the young Volterra in 1887, to which, under the influence of Hadamard, has been attributed an exaggerated historical importance.

More specifically, in connection with Volterra's notion of the derivative of a functional, Dieudonné further states:

> With our experience of 50 years of functional analysis we cannot help feeling that without even the barest notion of general topology, these ad hoc definitions were decidedly premature. [ibid]

(One might ask whether the pioneers of the calculus of variations were also 'premature'.)

To lay the ground for his response to Dieudonné, Fichera proposes that

> the true historian must make the effort to shed himself of today's way of thinking and of all the experience he has acquired ... discarding all the superstructures which have arisen during the passage of the years... [9]

My own defense of Volterra will be perhaps a bit less modest than Fichera's. Like Fichera, I believe that the possibility for an eminent French mathematician to make such statements derives partly from a conceptual identification by mainstream mathematics:

Functional analysis = the study of topological vector spaces.

But I want to depart from the above definition of 'true historian' and rather re-visit past developments in light of today's problems in order to explain my conclusion that this narrow conceptual identification is one of the reasons for the 'difficili rapporti', and is one of the ideas which must be refined to improve those relations.

The authors of a recent very useful biography of Hadamard [20], while acknowledging Fichera's arguments, nonetheless describe as 'naive' the theory of analytic functionals studied by Fantappiè, Pellegrino, Haefeli, Succi, Teichmüller, Silva, Volterra, Zorn and others [23]. More specifically, they say that

> The naivety of Fantappiè's approach lies in the fact that he, as Hadamard before him, avoided the topology and assumed analyticity in the sense of his general theory, instead of using continuity.

That these mathematicians were not in fact naive about the role of topology is adequately demonstrated by the fact that a portion of their work was devoted to showing that analytic functionals are automatically continuous with respect to some notion of neighborhood. But more importantly, as I will try to explain in section III, the 'general theory ' is more powerful than commonly supposed.

The idea of the preponderance of a neighborhood structure in infinite dimensional spaces grew with the enormous work on partial differential equations which has been accomplished in this century, in particular the work revolving around the notion of 'a well-posed problem in the sense of Hadamard' [20]. That important guiding notion is as follows: In the solution of certain problems of elasticity, electromagnetism, etc., the solution itself may change only slightly if the boundary data and region of definition are themselves varied sufficiently slightly. Making this guiding idea precise apparently requires specifying a notion of neighborhood in the domain space of the solution operator. But the difficulty is that these notions of neighborhood are not unique. Indeed general topology is so general that one could trivially achieve 'well-posedness' by simply defining the neighborhoods in the domain space to make the principle true (assuming the specification in the other space is less problematic; but note that a solution operator is typically a section of another operator, in the opposite direction, which one may hope is continuous also.) Of course, the well-posedness principle is not so tautological; one intends that the choice of topology (or neighborhood notion) is natural with respect to a category of problems at hand.

Grothendieck remarked to me in 1981 that already around 1950 he had had great difficulty in persuading analysts that there can be more than one natural topology for the same family of linear spaces. This remark in effect presupposes that there is some more basic cohesive structure which can be the 'same' for various notions of neighborhood. Perhaps this situation could be schematically represented as

$$\begin{tikzcd} & \mathrm{LBT} \arrow[dd] \\ \mathrm{C} \arrow[ur, dashed] \arrow[dr, "\mathsf{F}"'] & \\ & \mathrm{LB} \end{tikzcd}$$

where the family F of basically-cohesive linear spaces is functorially parameterized by a category C of problems, and we consider liftings across the forgetful functor from basically-cohesive spaces which are moreover compatibly equipped with a given topology.

But what could such a more basic notion of cohesiveness amount to? A very strong candidate is bornology. In their 'Convenient Setting for Global Analysis' [17], Kriegl and Michor state clearly that

> the locally convex topology on a convenient vector space can vary in some range - only the system of bounded sets must remain the same.

Their choice of morphisms for the purposes of infinite dimensional differential calculus was based on experience referred to, for example, in the fundamental work [25] on the spaces of differential forms :

> Nous allons définir dans chacun de ces espaces, une notion *d'ensemble borné*, qui sera utile pour analyser la condition de continuité qui intervient dans la définition des courants. (de Rham 1953)

and in the important step [26] in the development of general differentiation:

> Je me suis persuadé que, pour cette généralisation, c'est la notion d'ensemble borné, plutôt que celle de voisinage (ou de sémi-norme), qui doit jouer un rôle essentiel. (Silva 1960)

Waelbroeck, who also participated with Silva in the 1960 Louvain meeting, published two articles in 1967 (in the first volume of the Journal of Functional Analysis) clearly establishing the correctness of these views of
de Rham and Silva concerning the intimate connection of bornology with smoothness.

But there is another notion of basic structure more directly relevant to the approximation procedures involved in 'solving' partial differential equations by computer: Several different topologies can give rise to the same notion of *convergent sequence*. This is directly involved in another traditional construction in functional analysis which does not fit into the monolithic 'topological vector space' mode; for example Carathéodory, in his 1935 treatise on 'The Calculus of Variations' finds appropriate to give, as the first topic on page one of chapter one, the definition of 'continuous convergence' for a sequence f_k of functions defined on a domain A in n-dimensional space:

> For every possible sequence P_1, P_2... of points which lie in A and which converge to P_0, the limit $\lim_k f_k(P_k)$ always exists and represents a definite number. (my translation) [4]

This sort of definition of a cohesive structure on a function space is in fact forced by the universal property of cartesian closure, when the 'universe' is a category of spaces whose cohesion is determined by convergence of sequences. This fact is explicit in Johnstone [16] but also in Fox 1945 [10], though the latter did not explicitly use the term 'cartesian closed category' (nor the term 'λ-calculus' which his Princeton colleague Church introduced at about the same time to signify an important symbolic aspect of cartesian closure.)

Like the idea of convergent sequences, bornology is applicable in both linear and nonlinear settings (indeed bornology in its own sequential version has a very simple topos incarnation which could be considered parallel to Johnstone's sequential-convergence topos). However, it is only in tandem with linearity that bornology suffices for some part of global analysis (via the notion of Mackey convergence (1945)). Yet the needed applications of global analysis to calculus of variations or continuum physics are usually nonlinear. Another unspoken presupposition of mainstream mathematics seems to be

> *nonlinear is a generalization of linear and hence more difficult.*

But there are important ways in which a nonlinear category can be simpler than the linear category of vector space objects in it. These ways imply, as I will attempt to explain, that Volterra's Principle of the passage from the finite to the infinite can be freed from the limitations described by Fichera, i.e., that the Principle does in fact have the potential to deal with such phenomena as non-closed linear subspaces or total continuity. [9]

Nonlinear categories of C^∞ and of analytic spaces and maps are central to global analysis. The key concept which yields analytic maps in the sense of the Volterra school, namely that a good map is one that takes good paths into good paths, was applied successfully to define also a category of smooth maps by Hadamard [13], Boman [1], Lawvere-Schanuel-Zame [18], and Frölicher [11], and is central to [17] . In the latter, Kriegl and Michor state at the outset that they will define the smooth maps to be those which take smooth paths to smooth paths and that everything follows from that definition.

II

What are the main turn-of-the-century events which gave rise both to the current rich mathematical development, as well as to these historical controversies? In this section I attempt to give a very succinct summary.

After 200 years of both pure development and extensive application of calculus, Betti in 1887 presented to the Accademia dei Lincei a series of three notes by Volterra under the title ‘Sopra le funzioni che dipendono da altre funzioni’ (Opere pp 294-314). In that same year Volterra also treated ‘le funzioni dipendenti da linee’ and then in 1889 wrote [27]

> mais les points ne sont pas les seuls éléments géometriques.....

He emphasizes that also curves and surfaces (and we might add, tangent vectors) in X are elements of a space X, and just as one considers functions of points one must also consider functions of these species of elements as ‘generalized’ functions on X. Nor was Volterra driven solely by esthetic considerations (as Hadamard in his 1943 *Psychology of Mathematical Invention* inexplicably asserts), for he explicitly had in mind dependencies in continuum physics, such as the dependence of interior temperature on the boundary distribution of temperature of a body and similarly of the interior displacement of a flexible surface on the boundary distribution of displacements. In these papers Volterra made explicit some basic problems, concepts, and kinds of results which still occupy researchers in global geometric analysis and its application to continuum physics; among the key properties of functionals which he established [22] was the local existence theorem which is now referred to as the Poincaré lemma on the exactness of the de Rham complex of sheaves.

In 1897 Hadamard sent from Bordeaux a note [14] which was read by Picard at the Zurich International Congress. This note proposed the explicit consideration of sets of functions as mathematical spaces in their own right. (Bourbaki [2] considers this as the moment in history at which the usefulness of set theory began to be accepted by the mathematical community; because of the nature of the problems to which Hadamard referred, we can see from today's perspective that sheaf theory, at least as much as the theory of abstract sets, was being called for.) Pincherle [24] rose at the Congress to point out that the sort of theory sought by Hadamard, involving cohesive variation within function space, was already well under development in Italy, not only in the cited works of Volterra, but also in works of Ascoli, Arzela, and others. By 1903 Hadamard had already mastered much of that theory and developed it further, in particular coining the term 'functional'; in 1910 he published, with the help of Frechet, his beautiful lectures [15] on the calculus of variations, including a detailed exposition of the warmly praised theory of Volterra. Among the many later developments, Volterra became widely known for his pioneering work on Boltzmann's hereditary elasticity, as well as for his treatment of the fluctuation of the fish population in the Adriatic, while Hadamard became famous, not only for his theory of 'well-posed' problems, but also for his analysis of Huygen's Principle concerning wave equations in spaces of an odd number of dimensions. In the subsequent period, the important concept of neighborhood in function space somehow came to be considered as all-important; I cannot fully account historically for that rigidification of ideas, though I hope someone will be able to do so.

III

In this last section I will try to summarize synthetically the mathematical situation as it appears in light of these and previous studies.

The core disciplines such as geometric measure theory and differential geometry (i. e. the domains of the main problems which are treated via partial differential equations) gave rise, in the work of Volterra and others of his epoch, to the great auxiliary subjects of algebraic topology and functional analysis. The typical infinite-dimensional spaces produced for study by those disciplines are nuclear or at least have all bounded sets precompact and hence are essentially disjoint from the Banach and Hilbert categories which are necessarily introduced in the course of numerical and other analyses of these spaces. Similarly, the smooth manifolds (which serve as domains of variation for these spaces) are essentially disjoint from the simplicial and combinatorial categories which are necessarily introduced in calculating their algebraic-topological qualities. An important foundational goal is the specification of a flexible category (or system of categories) which can serve as a setting for both these objective and necessary subjective aspects, and to account for the relation between them.

A category of locally convex topological vector spaces is often considered as the linear side of such a setting, with manifolds modeled on open subsets of these as the nonlinear side. However, this vast generality makes room for a whole menagerie of counterexample spaces which are much more complicated than either the nuclear space or Hilbert space aspects; moreover, the reliance on the contravariant structure consisting of neighborhood systems, in accounting for the needed cohesion, makes almost impossible the achievement of those intuitively simple function-space transformations which are expressed by the exponential law of cartesian-closed categories.

I have been using the vague term 'cohesion' (in the way that physicists used to employ the term 'continuity') to avoid prejudice in favor of one or the other determination of it. Certainly we are considering that 'C^∞-structure' is a central kind of determination, and that 'topology' in the now-standardized sense is an important derived determination. In general, we might have occasion to treat, as a species of cohesion, almost any *extensive* category [5] for the non-linear side, and for any given rig object R

in it, the category of all internal R-modules as a corresponding linear side; a linear subcategory of 'complete' R-modules can often be defined in terms of the necessary higher-order structure discussed below. But how are significant non-combinatorial examples, of extensive categories with cartesian products, to be constructed?

A paradigmatic example has been the consideration of sets equipped with neighborhood structures, with cohesion-preserving morphisms f considered to be those which *contravariantly* respect the neighborhood structure: "No matter how small a neighborhood V of f(x) is demanded, a neighborhood U of x can be found which is so small that $U \subseteq f^{-1}V$ ", i.e., that f^{-1} preserves openess. The trouble with this determination, as a basic setting, is that, as was pointed out by many authors, it does not support the supple higher-order structure which it was designed to model. Many proposed remedies (such as those of Kelley, Spanier, Brown, and Steenrod), although perhaps still couched in the language of neighborhoods, in effect replaced the structure-and-morphism specification by a *covariant* one such as "for any compact figure C in the domain of f , fC is a compact figure in the codomain".

Both bornology and convergence, by contrast, are inherently covariant and hence lead immediately to cartesian-closed categories in a manner quite painless compared with the restricting remedies which had to be imposed on contravariant structures to achieve a similar end.

In general, if in a category a subcategory of special 'forms of elements' is distinguished, then a general 'space' X in the category determines a geometric structure consisting of elements $A \longrightarrow X$ and incidence relations given by $A' \longrightarrow A \longrightarrow X$ for $A' \longrightarrow A$ in the subcategory; sometimes this covariantly-associated structure determines the spaces, in the sense that any abstract mapping $X \cdots\cdot \rightarrow Y$ of all these elements comes from an actual space morphism $X \longrightarrow Y$ if only it is compatible with the incidence relations. Dually, one could consider a subcategory of objects R as basic quantity types, and associate (in a contravariant manner) to each general space X an algebraic structure consisting of functions $X \longrightarrow R$ with algebraic operations given by

$X \longrightarrow R \longrightarrow R'$. Even when the algebraic structure does not determine the general space, it may provide useful notions of approximation; it is often possible to arrange that adequate 'forms of elements' A can be chosen so that the R-algebra structure suffices at least *for them.* Given any rig object R, with its multiplicatively-invertible part U, we define a subspace of a general space X to be 'R-open' if it is of the form $\phi^{-1}U$ for some function

$\phi: X \longrightarrow R$ in the category; then every morphism f in the category is automatically R-continuous! Note that all sober objects in the usual category of topological spaces are entirely determined by this 'algebraic' structure, with the choice R = the two-point Sierpinski space.

One could dispute the necessity for a cartesian-closed base category of spaces, if one were content to deal only with spaces of extensive quantities [19]. For example, on the category top of all topological spaces, there is the functor C_+ which assigns to every X the *abstract additive monoid* $C_+(X)$ of all nonnegative continuous real functions on X; this functor has an adjoint which composes with it to yield a linear monad M_+ on top, with

$\delta: X \longrightarrow M_+(X)$ a continuous map and $M_+(X+Y) = M_+(X) \times M_+(Y)$. A key concept of functional analysis, namely that of a continuous path of extensive quantities, is thus approached via maps $I \longrightarrow M_+(X)$ from the interval. I do not know exactly what are the Eilenberg-Moore spaces for this monad; they are in some sense 'complete' linear spaces. Of course, serious problems of analysis begin when one considers not-necessarily-positive quantities; but the abelian-group reflections of the M_+-spaces should provide an approach to that. This example was mentioned as an analogy in my 1966 Oberwolfach lecture in which I proposed the study of S-valued measures in the category Top of S-toposes; that category shares with top the feature of not being closed in general, but rather of admitting only 'locally compact' objects as exponents. But among the many striking results of the recent work by Bunge and Funk [3] is that, as in top, the Top-measures do enrich to become the points of new toposes in Top, even though the corresponding intensive quantities in general do not.

Both intensive quantities as well as extensive quantities (varying over spaces) need to be representable by spaces in order to fully exploit the possibilities of functional analysis in approaching the content of continuum physics. This leads inevitably to the basic requirement that the category of spaces be cartesian closed. Recall that this means that for any space A, there is for any space B an exponential space B^A characterized by the adjointness 'conversion' property that for any space T, the maps from T to B^A correspond naturally to the maps from $A \times T$ to B. If T ranges over an adequate subcategory of 'element forms', then the conversion property uniquely determines the various elements and incidence relations in B^A and hence determines B^A itself. If R is another space, a *functional* is then a map

$$B^A \longrightarrow R$$

Actually, we can distinguish two important kinds of functionals-functor. For fixed A and R, the maps $X^A \longrightarrow R$ determine a contravariant algebra of 'extended functions' on spaces X, which become Volterra's 'funzioni di linee' when A is one-dimensional. On the other hand, for fixed B (for example R) and variable exponent, the homogenous maps $R^X \longrightarrow R$ constitute a covariant functor of spaces X which represents the 'double-dual' approach to extensive quantities (as integration processes) associated with the name of F. Riesz. Of course, in any cartesian-closed category exponentiation can be iterated, so that the *spaces* of intensive or extensive quantities have a uniquely determined notion of variable element internal to themselves. As I pointed out in 1967, the Fermat-Study-Kähler approach to tangents and cotangents works also well in a C^∞ category, which is a natural enlargement [7] of the basic path-determined one: For a suitable infinitesimal element-form A, X^A is precisely the tangent bundle of X and the homogeneous 'funzioni di linee infinitesimali' $X^A \longrightarrow R$ are the differential forms. Combining these ideas, one represents de Rham's currents as maps of the kind $R^{\underline{X}} \longrightarrow R$ where $\underline{X} = X^A$ for infinitesimal A.

The fact that basic smooth categories are 'path-determined' seems a strong form of a principle of passage from the finite to the infinite of the kind possibly envisaged by Volterra. It means in particular that the whole smooth category is 'generated' in the sense that every object, including the infinite-dimensional spaces, is a direct limit of finite-dimensional spaces (among which are the element-forms such as a line); such a thing is surely false for any *linear* functional-analytic category, which is perhaps why Volterra's principle seems limited when confined to such a category. Using the (nonlinear) functions and curves, it is reasonable to define notions of 'R-closed part' Y of an infinite dimensional space X by requiring

(1) that Y be the equalizer of some pair of maps from X to R, or

(2) that the inclusion consider R as an 'injective object'; or

(3) that for all paths $R \longrightarrow X$ the inverse-image of Y be closed in R in the sense of (1) or (2).

(The second of these possibilities is in the spirit of the notion of R-opens mentioned above, but turns Tietze's theorem into a principle, as is implicit in the usual approach to algebraic geometry.)

It would seem that sufficient knowledge of functional analysis *and* category theory has been achieved in 100 (respectively 50) years to permit the formulation of a concrete basis (for global geometric analysis) which is at least approachable without first achieving the erudition of expertise in topological vector spaces. A rough outline is as follows. On the monoid of C^∞ endomaps on a one dimensional space R, there is the topos E_1 of sheaves; this is a locally-cartesian-closed (and locally extensive) category with excellent exactness which contains the Frölicher category F of [12] as a reflective subcategory; the reflector preserves finite products but not equalizers, so that F is not so exact and not locally-cartesian-closed although, crucially, it is globally cartesian closed. The virtue of F is (by definition) that every space in it is (like the line itself) equally well described in terms of line elements and their 'incidence' (=reparameterization of paths) or by the algebra of functions with finite-dimensional values. (Perhaps a reasonable parameterized version of this Frölicher duality axiom can be found which will yield a subcategory F(X) of the topos E_1/X with a reflector preserving finite products, thus after all achieving a reasonable modified version of local cartesian closedness.) Since by Boman the algebraic theory of C^∞ algebras is a full subcategory of E_1 and since the smooth toposes E_∞ considered in Synthetic Differential Geometry [21] may be construed to consist of (dual) sheaves on finitely-generated C^∞-algebras, there is a natural inclusion $E_1 \longrightarrow E_\infty$ of toposes which in a suitable sense generates E_∞ . The virtue of E_∞ is that it contains infinitesimal element forms A which even admit 'fractional exponentiation' $(\ )^{1/A}$ so that jets and currents become firmly representable. Strikingly, the exponential of infinitesimals A^A has the one-dimensional space R of homotheties (i.e. time-speed-ups or motion-retardations) as a retract, so that in a precise sense the cartesian-closed category E_∞ of global analysis is literally infinitesimally-generated! The internal R-homogeneous maps from R^X to R constitute a linear monad M whose Eilenberg-Moore objects are very 'complete' linear spaces; but all spaces of the intensive kind R^X or the extensive kind M(X) are automatically that complete. The natural appearance of M(X) makes it (and its ramifications) a central concept of extensive quantity perhaps replacing the idea of 'distributions of compact support' with which it agrees at least on finite-dimensional manifolds.

Within the linear categories so arising it is surely possible to discern ideals of the nature of compact operators or nuclear operators and indeed to analyze the concrete non-linear origin of such ideals in the ideals of strong inclusions (among the subspace inclusions within a domain space X). That structures existing in finite dimensions tend to persist in infinite dimensions need not be construed (as Fichera apparently did) to mean that statements (such as ‘all operators are nuclear’) should be expected to persist. That is, structure which is trivial may become non-trivial (for example the ring R^X has nonzero 2nd order linear differential operators if $X = R^n$, but not if $X = n$); but all structure is still describable in terms of its effects on finite parameterizations.

We can even attempt to describe a principle of passage from finite to infinite compatible with the additional century of experience. While we clearly need several categories and transformations between them (for example a simplicial topos and a real analytic topos mediated by a sequential-convergence topos) yet within each of the basic categories it is reasonable to expect that the following sort of construction is uniquely determined. Given two spaces A and B we can form B^A, and then the part P of B^A defined by an equation between two maps from B^A to C; then it makes unique sense to speak of a variation of a function on P, with the variation within the category. Stronger still, every space P is in fact determined by its elements (with incidence) of a few representable finite-dimensional forms (such as points, curves, surfaces, tangent vectors, 2 jets). This principle was reasonable 300 years ago, 200 years ago, and 100 years ago, and now, in much more explicit form, seems again reasonable and realizable, despite a century of counterexamples concerning contravariant cohesion, during much of which such a principle seemed ‘naive’.

Bibliography

[1] Boman, J. Differentiability of a function and of its compositions with functions of one variable, *Math. Scand.* **20**, 429-268, 1967.

[2] Bourbaki, N. *Eléments d’histoire des mathématiques,* Hermann, Paris, 44, 1960.

[3] Bunge, M., Funk, J. Spreads and the symmetric topos, *Journal of Pure and Applied Algebra* **113,** 1-38, 1996.

[4] Carathéodory, C. *Variationsrechnung und Differentialgleichungen erster Ordnung*, B.G. Teubner, Leipzig und Berlin, 1935.

[5] Carboni, A., Lack,S., Walters, R. Introduction to extensive and distributive categories, *Journal of Pure and Applied Algebra* **84**, 145-158, 1993.

[6] Dieudonné, J. *History of functional analysis*, Mathematics studies **49**, North-Holland, Amsterdam, 1981.

[7] Dubuc, E. J. C^{∞}-schemes, *American J. Math.* **102**, 683-690, 1981.

[8] Fichera, G. I difficili rapporti fra l'analisi funzionale e la fisica matematica, *Rendiconti dell'accademia nazionale dei Lincei,* **9** (1), 161-170, 1990.

[9] Fichera, G. Vito Volterra and the birth of functional analysis, *Development of Mathematics 1900 - 1950*, Pier, J.-P. ed., Birkhäuser, Basel, 1994.

[10] Fox, R. On topologies for function spaces, *Bull. Amer. Math. Soc.* **51,** 429-432, 1945.

[11] Frölicher, A. Applications lisses entre espaces et variétés de Fréchet, *C.R. Acad. Sci.* Paris **293**, 125-127, 1981.

[12] Frölicher, A., Kriegl, A., *Linear Spaces and Differentiation Theory,* J. Wiley, Kriegl A. Chichester, 1988.

[13] Hadamard, J. La notion de différentielle dans l'enseignement, *Script Univ. Jerusalem* **1**, 1923, reprinted in *Enseign. Sci* **35,** 136-137 (1931).

[14] Hadamard, J. Sur certaines applications possibles de la théorie des ensembles, *Verhandlungen des ersten internationalen Mathematiker Kongresses,* B.G. Teubner, Leipzig, 201-202, 1898.

[15] Hadamard, J. *Leçons sur le calcul des variations*, Hermann, Paris 1910.

[16] Johnstone, P. On a topological topos, *Proc. London Math. Soc.* **38** 237-271, 1979.

[17] Kriegl, A., Michor, P. W.*The Convenient Setting of Global Analysis*, American Mathematical Society **73**, 1997.

[18] Lawvere, F. W., Schanuel, S., Zame, W. On C^{∞}-infinity function spaces. (unpublished) 1981. summarized in [12].

[19] Lawvere, F. W. Categories of Space and of Quantity, *The Space of Mathematics. Philosophical, Epistemological and Historical Explorations.* DeGruyter, Berlin (1992) 14-30.

[20] Maz'ya, V., Shaposhnikova, T. *Jacques Hadamard, a Universal Mathematician.* History of Mathematics, American Mathematical Society/London Mathematical Society **14,** 1998.

[21] Moerdijk, I., Reyes G. *Models for Smooth Infinitesimal Analysis*, Springer-Verlag, New York, 1991.

[22] Narashiman R. *Analysis on real and complex manifolds*, 3rd ed. North-Holland, Amsterdam, viii-ix, 1985.

[23] Pellegrino, F. La théorie des fonctionnelles analytiques et ses applications, part 4 of Levy P., *Analyse Fonctionnelle,* 2nd ed., Gauthier-Villars, Paris 1951.

[24] Pincherle, S. Remarque relative à la communication de M. Hadamard, *Verhandlungen des ersten internationalen Mathematiker Kongresses,* B.G. Teubner, Leipzig, 203, 1898.

[25] de Rham, G. *Variétés différentiables*, Hermann, Paris, 1953.

[26] Silva, J. S. Les espaces à bornés et la notion de fonction différentiable, *Colloque sur L'Analyse Fonctionnelle,* Louvain 1960, Centre Belge de Recherches Mathématiques, 57-62, 1961.

[27] Volterra, V. Sur une généralisation de la théorie des fonctions d'une variable imaginaire, *Acta Mathematica* **12,** 233-286, 1889

Mathematics Department
SUNY at Buffalo
106 Diefendorf Hall
14214 Buffalo, N.Y. USA
e-mail: wlawvere@acsu.buffalo.edu June 1, 1998