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GRASSMANN'S DIALECTICS AND CATEGORY THEORY

PROGRAMMATIC OUTLINE

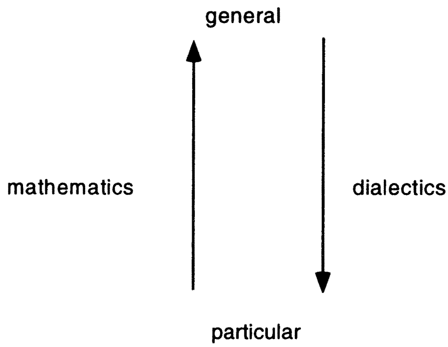
In several key connections in his foundations of geometrical algebra, Grassmann makes significant use of the dialectical philosophy of 150 years ago. Now, after fifty years of development of category theory as a means for making explicit some nontrivial general arguments in geometry, it is possible to recover some of Grassmann's insights and to express these in ways comprehensible to the modern geometer. For example, the category $\mathcal{A}$ of affine-linear spaces and maps (a monument to Grassmann) has a canonical adjoint functor to the category of (anti)commutative graded algebras, which as in Grassmann's detailed description yields a sixteen-dimensional algebra when applied to a three-dimensional affine space (unlike the eight-dimensional exterior algebra of a three-dimensional vector space). The natural algebraic structure of these algebras includes a boundary operator ∂ which satisfies the (signed) Leibniz rule; for example, if A, B are points of the affine space then the product AB is the axial vector from A to B which the boundary degrades to the corresponding translation vector: $\partial(AB) = B–A$ (since $\partial A = \partial B = 1$ for points). Grassmann philosophically motivated a notion of a "simple law of change," but his editors in the 1890's found this notion incoherent and decided he must have meant mere translations. However, translations are insufficient for the foundational task of deciding when two formal products are geometrically equal axial vectors. But if "law of change" is understood as an action of the additive monoid of time, "simple" turns out to mean that the action is internal to the category $\mathcal{A}$ at hand, in the following sense: The affine category has a unique closed structure, consisting of an internal hom functor with an adjoint tensor-product functor which neither is the cartesian product nor underlies the usual tensor product of vector spaces, and since addition of times does define an internal monoid structure $R \otimes R \to R$ with respect to this tensor product, it is very natural to interpret "internal law of becoming" on a space E to mean an action (= "flow") $R \otimes E \to E$ with respect to this tensor. Such actions

G. Schubring (ed.), Hermann Günther Graßmann (1809–1877): Visionary Mathematician, Scientist and Neohumanist Scholar, 255–264.

turn out to be determined by shear transformations, of which there are indeed enough to detect equality of axial vectors!

ORIENTATION

Grassmann in his philosophical introduction describes the two-fold division of formal sciences, that is, the science of thinking, into dialectics and mathematics. He briefly describes dialectics as seeking the unity in all things, and he describes mathematics as the art and practice of taking each thought in its particularity and pursuing it to the end. There is a need for an instrument which will guide students to follow in a unified way both of these activities, passing from the general to the particular and from the particular to the general.



I believe that the theory of mathematical categories (which was made explicit 50 years ago by Eilenberg and Mac Lane, codifying extensive work done by Hurewicz in particular during the 1930's), can serve as such an instrument. It was introduced and designed in response to a very particular question involving passage to the limit in calculating cohomology of certain portions of spheres, but this particular calculation necessitated an explicit recognition of the manner in which these spaces were related to all other spaces and, in particular, how their motion might induce other motions. In other words, category theory was introduced (and still serves) as "a universal geometrical calculus."

Looking more closely into Grassmann, Stephen Schanuel and I found numerous ways in which it could be justified to say that Grassmann was a pre-cursor of category theory. The general algebraic operations which he discussed have become the explicit object of a particular developed theory, and those general concepts, general operations, systems of operations and systems of equations in invariant coordinate free form have been made into a part of category theory. More specifically, we find that in certain cases the famous distinction between analytic and synthetic operations can only be explained in terms of adjoint functors.

This is because, although in simple cases the analytic operation may be the actual inverse of the synthetic one, there are a great many cases (arising in deductive logic and finance and so forth) where there are clearly pairs of operations having this analytic/synthetic relation, but the precise mathematical form of their relationship is not that of inverse, but rather of adjointness, in the sense of category theory. I hope to illustrate the use of category theory in making explicit certain of Grassmann's ideas, which have been misunderstood or neglected, with a few examples. I have written several other papers on the general principles of dialectics and how these can be formalized using category theory, so I think it is appropriate here to describe two particular algebraic examples worked out with Schanuel.

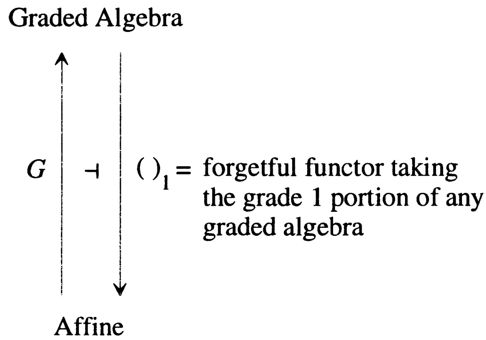
GRASSMANN FUNCTOR MORE STRUCTURED THAN EXTERIOR ALGEBRA

If as modern geometers we look into Grassmann's *Ausdehnungslehre,* we see on page 305 that the Grassmann algebra built on three-dimensional space has itself dimension 16, whereas in all the modern text books, the exterior algebra of a three-dimensional space has dimension 8. This distinction is by no means trivial; in fact a large part of the geometrical content of Grassmann's algebra, as he explains it in detail in the book, can be traced back to this opposition which stems from the opposition between points and vectors. As explained in most modern text books, the space of points has no preferred origin and the space of vectors lives principally because it acts on points; a special property of this action is that with two points there is a unique vector which is their difference and therefore the same translation motion by the vector can be applied to a new starting point.

The Grassmann algebra is the anti-commutative graded algebra which is universally associated to the space. There is the category of "commutative" (which is the modern word for anti-commutative) graded algebras and homomorphisms thereof. In particular, there is the possibility of taking the grade 1 portion of any such algebra which we can consider as an affine space of points; moreover, any homomorphism which preserves the grading and the addition and multiplication between these graded algebras will induce an affine linear map between the grade 1 portions. Now, the algebra Grassmann considered, (and discussed in detail the nature of grade 0, grade 1, grade 2, grade 3, grade 4) is uniquely characterized as the construction which is left adjoint to this forgetting process. This is one of the qualitative improvements of category theory over previous formulations, that a process can have a uniquely determined reverse attempt, which is however not actually an inverse.

Thus the Grassmann construction is uniquely determined by the fact that it assigns to any affine space of points a commutative graded algebra with the original points included into the grade 1 portion of the algebra in a way which is universal with respect to homomorphisms of graded algebras.

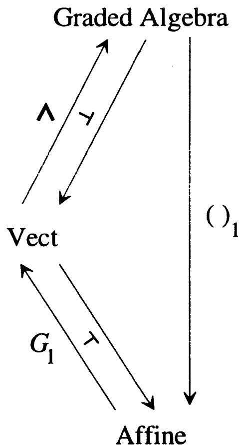
⊣ means "left adjoint of"



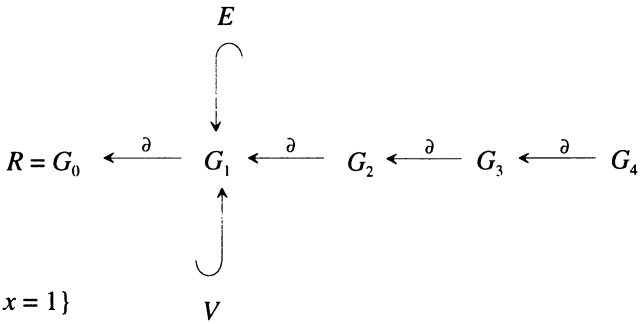
We can explain this construction with the help of the standard 20th century constructions by noting that this forgetful (throwing away the multiplication and higher grades) functor can actually be factored into two parts, namely the grade 1 portion of an algebra is a vector space and then we can forget the origin. If we take a vector space and forget the origin, what's left is an affine space. Now, it's a general principle of adjoints (proved by Kan in 1958) that the adjoint of a composite is the composite of the adjoints. So the Grassmann adjoint G is actually the composite of two left adjoints: the second is the exterior algebra of a vector space, as described in the text books; in particular, it assigns to a vector space of dimension n a graded commutative algebra of dimension 2^n; for example, to a three-dimensional vector space it associates an 8-dimensional algebra, and to a four-dimensional vector space it associates a 16-dimensional algebra etc.

Λ = modern exterior algebra

$G = \Lambda \circ G_1$



The process of forgetting the origin is not trivial, because the left adjoint to it is the following construction: You have a flat space (another word for affine is "flat") and you must adjoin in a universal way an origin. That origin cannot be in the original space, it must be adjoined outside, so the vector space generated by the original affine space and this new origin will always be of dimension 1 more than the dimension of the affine space. For example, the 3-dimensional affine or flat space by this composite process has at first associated to it a 4-dimensional vector space; so if we then take the exterior algebra in the modern sense of that 4-dimensional vector space, we will obtain the 16-dimensional Grassmann algebra, the one that he describes step by step in his book.



$E = \{x \in G_1 \mid \partial x = 1\}$

$V = \{x \in G_1 \mid \partial x = 0\}$

It has a grade 0 portion which is just the scalar field R ; it has as grade 1 portion the 4-dimensional vector space G_1 generated by the 3-dimensional affine space E , a 6-dimensional grade 2 portion and at the end a 1-dimensional grade 4 portion. An extremely important feature of this construction is that the Grassmann algebra has much more *natural structure* than the exterior algebra. The natural structure (in the technical sense of my 1963 doctoral thesis) of any functor consists of all natural operations, where a natural operation is an assignment to every value of the functor of an operation which commutes with all the morphisms which are values of the functor. In this case the natural structure includes a "boundary" operator ∂ which decreases grade by one and commutes with all the maps induced by an arbitrary affine–linear map α .

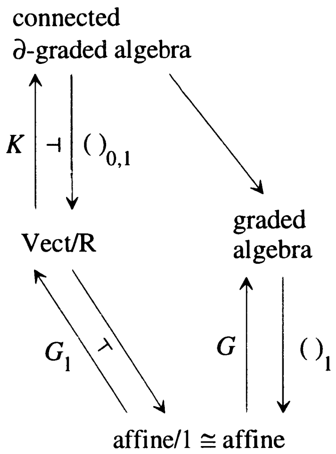
$$\begin{array}{ccccc}
E & & G_k(E) & \xleftarrow{\ \partial_E\ } & G_{k+l}(E) \\
\Big\downarrow \alpha & & \Big\downarrow G_k(\alpha) & & \Big\downarrow G_{k+l}(\alpha) \\
F & & G_k(F) & \xleftarrow{\ \partial_F\ } & G_{k+l}(F)
\end{array}$$

Thus we can consider the richer category of *differential* graded algebras and we find that the Grassmann functor actually lifts up along the forgetful functor (from this richer category back to the graded algebras). The axioms satisfied by the boundary operator in each differential graded algebra are that it is linear and satisfies the graded (signed) Leibniz rule with respect to the product, while of course reducing the grade by 1, and has square zero. By contrast, there is no such natural boundary operator on the exterior algebra of a vector space. Again we can explain this by modern constructions. The specific construction of the first step can be understood in the following way: The $G_1(E)$ is the 4-dimensional vector space generated by the 3-dimensional affine space E. There is the unique affine map from any affine space E to the one point affine space and yet $G_1(1) = R$. Hence, by the functoriality of G_1 there is a canonical linear map from $G_1(E)$ to $G_1(1) = R$; in fact, this map is just the lowest-grade instance of the boundary operator. If the elements of $G_1(E)$ are thought of as weighted points, then this linear functional simply specifies the weight value. The kernel of this linear functional is a 3-dimensional vector space which is actually the space of translations of E acting as follows: The part of $G_1(E)$ where this functional takes the value 1 is just the original affine space E , and obviously the addition of an element of value 0 takes elements of value 1 into elements of value 1 by linearity. Similarly, subtracting two points, that is, two elements of value 1, will give an element of value 0, i.e. a translation vector. The whole chain complex, i.e. the sequence of boundary operators of higher grade, can be described as the 20th century construction of the "Koszul" complex, the unique differential graded algebra structure which extends this lowest-grade part, which for the Koszul construction could be any linear functional on any given R-module.[1]

[1] For a non–empty affine space the homology of this complex is trivial, since a contracting homotopy is defined by the Grassmann product with any chosen point.

Vect/R = category of vector spaces equipped with a linear functional

K = Koszul



This boundary operator contains much of the geometry which Grassmann was talking about: for example, if we take two points, (i.e. two elements of G_1 which have boundary one) and form their Grassmann product, we obtain an element of G_2 which represents the axial vector from the first point A to the second point B. The boundary of the product is equal to $B - A$, the ordinary vector which translates A to B for

$$\partial (AB) = (\partial A)B \pm A\partial B$$

but if A and B are points, then ∂A and ∂B are both 1 and moreover in this grade it is the minus sign which we must take in the Leibniz rule. Higher products represent triangles, tetrahedra and so on and their boundaries are their usual combinatorial boundaries. Thus not only do the volume elements etc. live at the expected grades, but moreover, their boundaries are connected in the expected way.

Grassmann's term for the boundary operator is "Ausweichung." The exactness (vanishing of homology) he affirms on page 187; the special case, that the top-grade instance of the boundary operator is injective, is used for example on page 198. The fact that the lowest-grade instance of the boundary operator coincides with the weight functional is explicit on page 185. The quantities of boundary zero (which Grassmann called "Ausdehnungsgrösse") constitute the exterior algebra of the translation vector space of the original affine space; through their inclusion into the larger algebra these quantities can act on the "rigid" quantities which contain point factors.

"SIMPLE" LAWS AND EQUALITY OF AXIAL VECTORS

Grassmann apparently intended that his philosophical notion of "Becoming according to a simple law of change"[2] should serve as a foundation for his geometrical algebra. But Engel and Study, the editors of his collected works, found his discussion incoherent and decided that "in fact" (page 405) his changes were merely parallel translations. However, let us consider the foundational problem in the following sense. While indeed the description of the Grassmann algebra construction as the left adjoint into graded "commutative" algebras is very effective for calculation, we can ask why that particular category is chosen as geometrically natural. The crucial ingredient in presenting some algebra generated by points is the relation stating when two formal products of generating points are to be considered equal; there should be a criterion for $AB = CD$ in terms of the basic affine category itself. One idea for solving the foundational problem of providing such a criterion is as follows. Assuming that "simple law of change" is somehow interpreted as a special kind of affine–linear morphism, we can associate, with each pair of points A, B in a given affine space E, the set of all simple endomorphisms S of E for which $SA = B$; then define an equivalence relation on the set of pairs of points by the requirement that (for B distinct from A, and D distinct from C), these sets should be equal, in other words (A,B) is equivalent to (C, D) if and only if for every simple endomorphism S, $SA = B$ iff $SC = D$. If "simple" merely meant translations, this idea would collapse, because the equivalence would just say $B - A = D - C$, equality of the mere boundaries of the axial vectors. But it seems that Grassmann's description on the cited pages supports a broader interpretation of "simple" which both narrows the above equivalence idea enough to separate distinct bound or axial vectors and is also extremely natural in view of the particular nature of the affine category.

For an affine–linear endomorphism S of E, the geometrical property that for every point A, the product $A(SA)$ is the same as the product $(SA)(SSA)$, is equivalent to the algebraic property $S^2 = 2S - 1$. That equivalence can be proved in one direction by applying the boundary operator and in the other direction by using (anti)commutativity and the fact that the square of any point, such as SA, is zero. The foundational significance of this equivalence is that our "algebraic" condition does not presuppose the Grassmann product, but only involves composition and the fact that affine combinations of such maps (specifically the one with coefficients 2, –1) can be formed. Let us call such endomorphisms S "simple." Translation by a given vector is indeed simple, but there are

2 A_1, Einleitung, Nr. 9-10, and: Abschnitt 1, Kap. 1, §13 and §14, and: Anhang III, I. 5.

more examples such as any linear transformation which differs from the identity by a linear transformation of square zero, and certain combinations of these two types.

The simple laws of change in the above sense do indeed determine relations for presenting the Grassmann algebra. More precisely, not only does $AB = CD$ imply that any simple change taking A to B also takes C to D, b ut also that the converse is true. That converse can be proved as follows: the two sets of simple maps assumed equal include the translation and so, in particular, $D = C + (B - A)$, but more, if C were not on the line spanned by A and B, then there would be a simple S for which $SA = B$ but SC is not D (for example we could even have $SC = C$, a shearing transformation with C as origin.)

The apparently bizarre equation $S^2 = 2S - 1$ is in fact determined by applying to the affine category $\mathcal{A}$ the following very general consideration. It is a closed category in a unique way. This means that there is an internal hom functor with a left adjoint tensor product functor and unit object. For example, the category of sets has the exponential function-set of any two sets, the left adjoint to that is the ordinary cartesian product of two sets and the unit object is the one point set. The category of vector spaces has the well-known internal hom and tensor product whose unit is the ground field. More complicated categories may have more than one closed structure, but when the unit object is faithful, the closed structure is unique (but only rarely exists). For the category of affine spaces the unit object is the one point space, which is indeed faithful; the tensor product (uniquely determined by representing bi–affine maps) is bigger than the corresponding one for the category of vector spaces, since the functor (discussed previously) of adjoining a generic origin does transform the affine tensor product into the linear tensor product and hence $nm + n + m$ is the dimension of the affine tensor product of two spaces of respective dimensions n, m. In any closed category there is the notion of a monoid structure on an object M and of M–action on any object, both with respect to the tensor product. For example, in vector spaces a "monoid" is really any associative linear algebra and an "action" of such turns out to mean any module over the algebra. The affine tensor product of the scalar field R with itself is three-dimensional, and ordinary multiplication is an example of a monoid structure on R ; however, addition gives a different monoid structure on R in the affine category, which is the one that interests us here because actions of the additive monoid of time translations are considered in many categories as a model of "laws of Becoming" (or in modern terminology "dynamical systems"). Of course, when the monoid structure is written additively, the associative law required for any action looks like an exponential law (also sometimes known as the "semigroup property"). A bi–affine action of the additive monoid R of all real numbers on a given affine space E is determined by its instance at $t = 1$ because for other times it just affinely follows the straight line through its state A at $t = 0$ and its state SA at $t = 1$. The associative (or exponential) law is a strong restriction on which affine–linear endomorphisms S can occur within such a flow: they are exactly those satisfying $S^2 = 2S - 1$ which we interpreted above as Grassmann's simple laws of change. In this sense the relation imposed on the Grassmann multiplication has a foundation in the way in which the affine category particularizes the very general notion of internal law of change.

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