Adjoints in and among Bicategories

F. William Lawvere

Abstract

Professor Roberto Magari is remembered for the depth and clarity of his lectures which impressed students and colleagues alike. Indeed he championed the deepening of logic through algebraic models and the clarification of algebra through explicit recognition of logic. In that spirit, I here model the basic construction of proof theory as the left bi-adjoint to the inclusion of Posets into Categories. That simple description has a rich history, some of which I here recall. Its specific content stems from the analysis, within such bicategories, of quantifiers and implications as cases of Kan's notion of adjoint. The construction leads to some open algebraic problems concerning known classes of locally-cartesian-closed categories and their role as models for proof theory.

1 Some Basic Concepts Concerning Bicategories

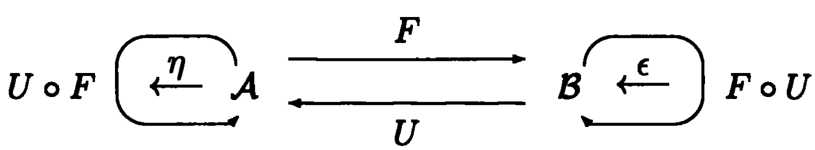
A bicategory is (apart from technical considerations concerning the precise sense in which composition is associative) a category enriched in Categories; that is, between any two objects there is (not just a class, but) a *category* of morphisms (whose objects are called 1-morphisms and whose morphisms are called 2-morphisms of the bicategory) and composition is a *functor* on the cartesian product category (for each triple of objects, or "0-morphisms", of the bicategory). Combinatorially, if we consider an ordinary category to be a multiplicative graph, where a graph is a conglomerate of 1-balls (arrows) with boundary 0-spheres given as the union of two points, then analogously, a bicategory is a bimultiplicative bigraph, where a bigraph is a conglomerate of 2-balls ("lozenges") with boundary 1-sphere given as the union of two 1-balls intersecting in *their* boundaries; the "vertical" multiplication of 2-morphisms is that within each hom-category, whereas their "horizontal" multiplication is the functorial composition of the bicategory. Just as category theory can explicitly encapsulate much more mathematics than pure set theory, while yet remaining universal, so bicategories contain qualitatively more information than pure categories. On the other hand, the notion of tricategory (and even ∞-category), which has proven useful in homotopy theory, has the striking feature that even there the concept of bicategory is central, since it is the structure relating *any two* levels. Explicitly, the functorality of composition implies the law

$$(ab) \circ (xy) = (a \circ x)(b \circ y)$$

involving three 0-morphisms, six 1-morphisms, and four 2-morphisms a, b, x, y which are composable, where juxtaposition denotes vertical composition and circle denotes the horizontal composition. The "bifunctors", which are the "morphisms" between bicategories, are of course required to preserve the bigraph structure as well as the two compositions (but again there is considerable content in the precise sense to which the composition of 1-morphisms is preserved, since it may be true only "up to an invertible 2-morphism").

2 Cat and Adjointness

A basic concrete example is Cat, where categories, functors, and natural transformations are the 0-morphisms, 1-morphisms, and 2-morphisms respectively. Indeed, as Yoneda's lemma permits analysis of ordinary categories in terms of Set-valued functors, so Yoneda's bilemma permits analysis of general bicategories in terms of Cat-valued bifunctors. That applies in particular to the notion of adjointness. It seems that a bicategory is precisely the most general environment in which the notion of adjointness can be defined, by any of the following three equivalent conditions on a pair of 0-morphisms, a pair of 1-morphisms, and a pair of 2-morphisms in the configuration



in which the two "lozenges" η and ϵ are squashed because half the boundary of η is reduced to the point $1_{\mathcal{A}}$ and the other half of the boundary of ϵ is reduced to the point $1_{\mathcal{B}}$. One condition is just that the two equations

$$\begin{aligned}(\epsilon \circ F)(F \circ \eta) &= 1_F \\ (\eta \circ U)(U \circ \epsilon) &= 1_U\end{aligned}$$

hold in the bicategory itself; these imply that, among the 2-morphisms (constructible from the data) between the iterated loops at $\mathcal{A}$ or $\mathcal{B}$, all the "simplicial" identities (familiar from composition of order-preserving maps between finite totally-ordered sets) hold. This can be Yoneda-tested as follows: for any $\mathcal{T}$ in the bicategory, the induced functors

$$Hom(\mathcal{T},\mathcal{A}) \underset{U}{\overset{F}{\rightleftarrows}} Hom(\mathcal{T},\mathcal{B})$$

between actual categories should be adjoint, with adjunction unit and counit induced by η and ϵ. That in turn has meant for over 35 years that for each pair A, B of 1-morphisms $\mathcal{T} \longrightarrow \mathcal{A}$, $\mathcal{T} \longrightarrow \mathcal{B}$, η and ϵ induce natural bijections

$$\frac{F \circ A \longrightarrow B}{A \longrightarrow U \circ B}$$

between the indicated sets of 2-morphisms. The third equivalent condition is that for each $\mathcal{V}$ in the bicategory, there is a similarly induced pair

$$Hom(\mathcal{A},\mathcal{V}) \underset{F^*}{\overset{U^*}{\rightleftarrows}} Hom(\mathcal{B},\mathcal{V})$$

of adjoint ordinary functors.

It is the first, equational, condition which makes it obvious that all bifunctors preserve adjointness. That is a far-reaching generalization of the fact that ordinary functors preserve isomorphism, just as the theorem that adjoint 1-morphisms uniquely determine each other is a far-reaching generalization of the fact that in an ordinary category inverse isomorphisms uniquely determine each other. Here the sense of uniqueness for 1-morphisms appropriate in a bicategory is "up to an invertible 2-morphism". The explicit sense in which the two mentioned results are generalizations stems from the construction, for each ordinary category, of the bicategory which extends it by taking 2-morphisms to be just equalities between the given 1-morphisms.

3 Posets and Quantifiers

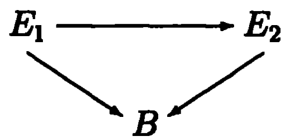
A key bicategory for predicate logic is the one in which 0-morphisms are posets, 1-morphisms are order-preserving maps, and wherein there is a 2-morphism from f to g if and only if $fx \leq gx$ in the (common) codomain for all x in the (common) domain. This bicategory is seemingly somewhat "trivial" in that any two 2-morphisms with the same domain f and the same codomain g

are themselves equal. Nonetheless, there are many non-trivial adjoints *in* it, such as the upper and lower integrals of the Darboux-McShane theory, as well as the semantic correspondents of the rules of inference usually considered in predicate logic. Indeed, these examples are of a special nature made possible by the function-space adjoint which acts *on* this bicategory of posets to give it its cartesian closed character. Explicitly, if T is any complete poset and $f : A \longrightarrow B$ is any order-preserving map, then there is an induced order-preserving map T^f which substitutes f into the various "predicates" in T^B yielding new predicates in T^A. This T^f has both a left adjoint and a right adjoint, which is the meaning of existential and universal quantification.

Often the term "quantifier" is restricted to the case of the adjoints induced by some special f's, as in first-order single-sorted predicate logic where $A = X^W$, $B = X^V$, $f = X^\sigma$, for maps $\sigma : V \longrightarrow W$ between finite sets of "variables"; even here the surjective (but not injective) σ's (so the injective f's) incorporate in their induced adjoints the rules of inference for equality predicates, whereas it is the injective (but not surjective) σ's (so the surjective f's) which induce the quantifiers which make a genuine existential or universal leap. However, most of the work on algebraic logic in the past twenty-five years has followed the more liberal interpretation according to which the operation of forming the image of definable parts of the domain of *any* term f is considered as an existential quantification, because it satisfies the appropriate rule of inference.

These two key features of the desired semantics of first-order predicate logic, namely the mutual determination of substitution and quantifications via adjointness and the role of the category of finite sets as variables, should be retained in any abstract algebraic structure which makes the universe X generic, and even in a syntactical scheme for presenting examples of such algebraic structures by generators ("atomic predicates") and relations ("axioms"). The early attempts in the 1950's to describe such algebraic structures in the form of cylindric or polyadic algebras dealt instead with the infinitude of variables by using a single infinite set, rather than the category of finite sets. This distortion of the intended content obscured the adjointness between substitution and quantification, which was left implicit and accounted for instead by special axioms which seemed peculiar to the subject, since they were at best vaguely analogous to conditions appearing in analysis, topology, or other parts of algebra. This double distortion of the content, as I pointed out at the 1963 Berkeley Model Theory symposium, meant that these two particular algebraic versions of logic were destined for relatively little application to mathematics.

Of course, the predicates $X^V \longrightarrow T$ (especially in case $T = 2$) classify certain subobjects of X^V or "relations" and the ordering given by the 2-morphisms reflects the objective inclusions between these subobjects; it is these objective inclusions and their transformations which are the content of "logic in the narrow sense", reflected subjectively as entailment relations between the formulas which name the subobjects. But inclusions between subobjects of B are established by commutative triangles



where the vertical arrows specify the insertions of the subobjects. Of course, since insertions of subobjects are monomorphisms, there is at most one $E_1 \longrightarrow E_2$ establishing the inclusion of any two given subobjects. On the other hand, in topology, geometry, combinatorics, etc. there often arises the need to compare $E_i \longrightarrow B$ which are not necessarily monomorphisms; this suggests the need for "logic in a broader sense" in which the attributes of B form a category which does not reduce to a poset. The text of my talk "Category-valued Higher-order Logic", at the 1967 Los Angeles Set Theory Symposium, was distributed to the participants of that Symposium and discussed also in the 1968 Versailles meeting on Automatic Demonstration (SLNM 125). Revised and expanded, that material was published in three papers ([3],[4],[5]). These three papers helped to popularize the new field of "proof theory", which had emerged in the work of H. Läuchli, circulated in Spring 1967 as a new complete semantics for intuitionistic logic. (The emergence had been predicted by an old remark of Curry concerning the striking analogy between modus ponens (for propositions and inferences) and the laws of functionality (for types and terms); that analogy had been objectified by my observation that both involve cartesian-closed categories ([1])). Läuchli used actual mathematical objects (namely permutation representations), rather than the ritual equivalence classes of strings of symbols, so his work was described at Versailles as "non constructive" and not seriously looked at again until very recently. The construction by D. Scott in 1970 of models for the untyped λ-calculus, using the remarkable properties of adjoint retractions in the cartesian-closed bicategory of posets with filtered sups, continued to popularize the virtues of examples constructed instead by means of inverse and direct limits from spaces; these virtues already manifested themselves in a remarkable way in the 1966 Scott-Solovay Boolean-valued models for set theory, which were seen to constitute an important corner of another bicategory, that of toposes.

4 The Curry-Läuchli Adjoint

The central problem of proof theory is the unbounded nature of "there exists a proof". Like any existential quantification, this concerns a left adjoint, but one embedded in a definite context. The essential content of this Curry-Läuchli adjoint is, I believe, independent of syntactic presentations and embodied in the following

Proposition. *The inclusion of the bicategory of posets into the bicategory of categories has a left adjoint which is a bifunctor (hence preserves adjointness) and which preserves finite cartesian products.*

Explicitly, the poset associated to a given category has as elements the objects of the category, and $A \leq B$ iff there exists a morphism $A \longrightarrow B$. Although this Curry-Läuchli adjoint has many applications, we pass immediately to the most typical. Recall that a category $\mathbf{C}$ is called locally cartesian-closed if it has pullbacks and if for each $A \to B$ in $\mathbf{C}$, the pullback functor f^* has a right adjoint $f\Pi$.

$$\mathbf{C}/A \begin{array}{c} \xrightarrow{\quad f\Sigma \quad} \\ \xleftarrow{\quad f^* \quad} \\ \xrightarrow[\quad f\Pi \quad]{} \end{array} \mathbf{C}/B$$

Here the left adjoint $f\Sigma$ is the functor which exists trivially for any category and which comes about by merely composing f with each $E \to A$ in $\mathbf{C}/A$; in case $B = 1$, the effect of $f\Sigma$ is to extract the total of an object distributed over A. The composite functor, $f\Pi$ following f^*, is the function-space construction which shows that each category $\mathbf{C}/B$ is cartesian closed.

Corollary. *If $\mathbf{C}$ is a locally-cartesian-closed category, denote by $\mathcal{P}_C(A)$ the poset reflection of $\mathbf{C}/A$, for each "type" A in $\mathbf{C}$. Then each $\mathcal{P}_C(A)$ is a Heyting algebra, in the sense that it has conjunction and implication related by modus ponens (as well as disjunction in case $\mathbf{C}$ has coproducts). For each "term" $A \to B$ in $\mathbf{C}$, f^* induces a substitution which is a Heyting homomorphism and has quantifiers*

$$\mathcal{P}_C(A) \begin{array}{c} \xrightarrow{\quad f\exists \quad} \\ \xleftarrow{\quad f^* \quad} \\ \xrightarrow[\quad f\forall \quad]{} \end{array} \mathcal{P}_C(B)$$

satisfying the correct rules of inference. Moreover, to give a "proof" in $\mathbf{C}/B$ of $(f\exists\varphi) \cdot (y)$ is to give an x for which $fx = y$ together with a proof of $\varphi \cdot (x)$, while to give a proof of $(f\forall\varphi) \cdot (y)$ is to give, for each x with $fx = y$, a proof of

$\varphi \cdot (x)$, ***but the latter in a uniform manner in the sense that it is implemented by a single map in C.***
[Notation: $f^*\varphi = \varphi \cdot f$, for example $\varphi \cdot (x) = x^*\varphi$, so that $(\varphi \cdot f) \cdot x = \varphi \cdot (fx)$. Here the x and y denote arbitrary C-morphisms from an arbitrary test object to A and B respectively. Note that $(gf)\exists\varphi = g\exists(f\exists\varphi)$ and that if for the case $f = X^\sigma$ we write instead $f\exists\varphi = \varphi\exists\sigma$, then we have $\varphi\exists(\sigma\tau) = (\varphi\exists\sigma)\exists\tau$ for the iterated quantifiers.]

In case **C** is the category of (small) abstract sets, the above construction recovers the usual power sets. but the situation is strikingly different for most Grothendieck toposes, where the axiom of choice is false. For a precise comparison, let $\Omega_{\mathbf{C}}(A)$ denote the poset of all **C**-subobjects of A. Then the epi-mono image factorization yields a retraction functor $\mathbf{C}/A \longrightarrow \Omega_{\mathbf{C}}(A)$, which by the universal property of the poset reflection yields a retraction of posets

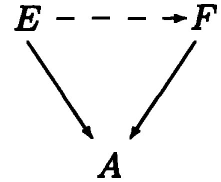
$$\mathcal{P}_{\mathbf{C}}(A) \longrightarrow \Omega_{\mathbf{C}}(A).$$

The latter is an equivalence for each A iff all epimorphisms in **C** split; otherwise $\mathcal{P}_{\mathbf{C}}(A)$ is bigger. In fact, it can be a proper class, as for example if **C** is the topos of graphs, though Läuchli's work showed that it is small in the case where **C** is the topos of permutation representations of a given group.

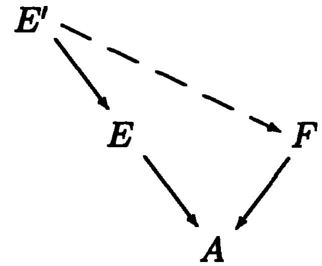
5 Open Problems

It seems to be still an open question to determine for which Grothendieck toposes **C** the "proof-theoretic power set" $\mathcal{P}_{\mathbf{C}}(A)$ is always a small Heyting algebra.

There is a natural weakening of the stringent requirement

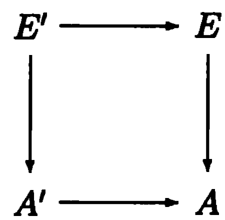


for the provability of "E implies F". Namely, we could first analyze the hypothesis by passing to a judiciously-chosen epimorphic cover $E' \longrightarrow E$ (which will have the same subobject of A as image) and then try for a direct proof of the conclusion from the refined hypothesis:



This turns out to be equivalent to the existence of an actual inclusion between the image subobjects in $\Omega_{\mathbf{C}}(A)$!

In a topos the subobjects of A are internally representable by maps $A \longrightarrow \Omega_{\mathbf{C}}$ to a fixed representing object. To achieve something like that for $\mathcal{P}_{\mathbf{C}}(A)$, we would have to replace $\mathcal{P}_{\mathbf{C}}$ by its sheafification and to verify that at least that sheaf is small. This sheafification involves a special form of the "refinement of hypotheses" idea of the previous paragraph, namely passing first to a covering of the universe A and then pulling back the hypothesis:



For which Grothendieck toposes is the sheafification of the proof-theoretic power set small, and hence internally representable?

Since directed graphs are actually diagrams of a finite shape, the combinatorial aspect of these open questions is underlined by the special case: for which finite categories does the topos of presheaves have the property that the proof-theoretic power set functor has small values?

6 The Special Role of Negation

The proof theory of negation seems at first problematic since in any cartesian-closed category with an initial object "false", any object of the form $(E \Rightarrow \text{false})$ is already in the poset of subobjects of 1, seemingly obliterating any possibility of making distinctions between proofs of a negative formula. That Läuchli nonetheless obtained a completeness result for intuitionistic predicate calculus (in spite of using only Boolean toposes in his semantics) stems from his understanding the negation of an object E in $\mathbf{C}/A$ not merely as another object, but as the functor of F in $\mathbf{C}/1$

$$(\neg E)(F) = (E \Rightarrow A^*F)$$

where A is the map from A to 1. The same idea would seem relevant in contexts other than proof theory. For example, in any topos, negation so defined becomes a monomorphism from truth-values to unary propositional operators. Similarly, in the "generalized logic" over any metric space A, a Lipschitz function E is not recoverable from its zero-set alone, but is recoverable from its family of superlevel sets, one for each constant A^*F for F real.

References

[1] S. Eilenberg and G.M. Kelly, *Closed Categories*, Proceedings of the Conference on Categorical Algebra, La Jolla 1965, Springer Verlag, 1966.

[2] H. Läuchli, Manuscript circulated in Spring 1967.

[3] F.W. Lawvere, *Adjointness in Foundations*, Dialectica **23** (1969), 281–296.

[4] ———, *Diagonal Arguments and Cartesian Closed Categories*, Lecture Notes in Mathematics **92**, 1969, pp. 134–145.

[5] ———, *Equality in Hyperdoctrines and the Comprehension Scheme as an Adjoint Functor*, Proceedings of the AMS Symposium on Pure Mathematics, XVII, 1970, pp. 1–14.

[6] ———, *Metric Spaces, Generalized Logic, and Closed Categories*, Rend. Sem. Mat. Fis. Milano **43** (1973), 135–168.

[7] M. Makkai, *The fibrational formulation of intuitionistic predicate logic: Completeness according to Gödel, Kripke and Läuchli*, Notre Dame J. Formal Logic **34** (1993), 334–337 and 471–498.

Department of Mathematics, SUNY at Buffalo, 106 Diefendorf Hall, Buffalo, NY 14214, USA

e-mail address: MTHFWL@UBVMS.CC.BUFFALO.EDU