Cohesive Toposes and Cantor's '*lauter Einsen*'

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Some years ago I began an introductory course on Set Theory by attempting to explain the invariant content of the category of sets, for which I had formulated an axiomatic description. I was concerned to present an ideological vision of the significance of the objects of this category, which I called *abstract sets*. I emphasized that an abstract set may be conceived of as a bag of dots which are devoid of properties apart from mutual distinctness. Further, the bag as a whole was assumed to have no properties except *cardinality*, which amounts to just the assertion that it might or might not be isomorphic to another bag. After hearing this description, John Myhill (who attended the first few lectures) said to me: 'I have seen all this before.' 'Where?' I asked. 'In Cantor', came the reply. Later he brought me his copy of Cantor's works with a note saying: 'See page 283 where Cantor speaks of "...*lauter Einsen*".' (This unusual German expression could mean roughly 'nothing but many units'.) Cantor speaks on the one hand of '*Mengen*' and, on the other, of '*Kardinalen*'. Myhill had noticed that Cantor's description of '*Kardinalen*' and my description of 'abstract sets' were essentially the same.

The term '*Mengen*' is normally translated as 'sets', and of course every book on set theory contains copious references to 'cardinals', but these 'cardinals' are completely different from those described by Cantor himself. Moreover, '*Mengen*' in Cantor's sense are not explicitly discussed at all in these books. They must, of course, be implicitly present in order to justify the claim that 'set theory' has a genuine relationship with mathematics, but as theoretical objects in Cantor's sense they play no role. It seems then that we, the mathematicians of this century, have neglected an important component of Cantor's concept of set. How did this come about?

This omission may partly stem from overzealous 'guidance' on the part of editors of collected works of great nineteenth century mathematicians. It is natural, in the case of recondite original works, to follow the 'guidance' of a knowledgeable editor. But the result can be misleading. Zermelo, the

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editor of Cantor's works, asserts that, despite Cantor's greatness, his account of the passage to '*Kardinalen*' is incorrigibly inconsistent, and that in any case it is more important to proceed immediately to the arithmetic of cardinals—cardinals, that is, in the contemporary sense. More precisely, Zermelo says that this attempt to explain cardinals as the result of a process of abstraction involving the '*lauter Einsen*' was 'not a happy one', because these '*Einsen*' must be different from one another, but how can they be different if they have no distinguishing properties? This contradiction, which is really, as I will show, a contradiction in a productive sense, seems to have led Zermelo to conclude that the whole concept is inconsistent, that one cannot speak of cardinality in this manner, and, therefore, that if one persisted in doing so it would not be possible to move forward to interesting cardinality calculations—even though Cantor himself did exactly that!

Similar assertions of inconsistency and impossibility are footnoted to some of the most interesting passages in H. Grassmann's collected works by his editor E. Study. Thus when studying the works of the great mathematicians of the last century we must strive afresh to find the core content of their thought, without being prejudiced by the opinions of the editors of their collected works, and others during the period after the last decade of the century.

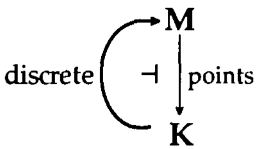
The word isomorphism has two kinds of meanings: First, in an actual category some maps in particular might be invertible; second, an equivalence relation among the objects is defined by the *existence* of isomorphisms in the first sense. While Cantor of course used the second abstraction too (as 'same cardinality'), he seems to have used the term *Kardinale* to denote a prior, more particular, abstraction in which an actual category of a more purified nature is extracted from a richer one, accompanied by specific connections between the two categories.

We know that the concept of equicardinality of '*Mengen*' is somehow concerned with a kind of isomorphism which in Cantor is called 'Mächtigkeit', that is, two '*Mengen*' are of equal potency if there exists a bijection between them. Now a very interesting point which emerges from a study of Cantor's works is that he himself cites the origin of the word 'Mächtigkeit' in the work of the great Swiss geometer Jakob Steiner, who apparently used this term to signify isomorphism in a *different* category, namely, the category of algebraic spaces. In his 1850 work on conic sections, he uses the concept of equal potency to explain how the ellipse is not equivalent to the parabola, nor the parabola to the hyperbola, in an intrinsic geometric sense. These are all objects in one simple category in which there are many different geometric objects, over which there is defined a concept of isomorphism whose invariants are just the usual geometric ones. Cantor himself asserts that he lifted this concept of isomorphism from its geometric context in order to arrive at *his*—necessarily more abstract—concept of isomorphism. This fact is not to be found in any book I have seen (although it is emphasized in a recent paper of Colin McLarty who noticed it independently). One can say that here the concept of category is already present in embryonic form, because there is an implicit unifying concept in the analogy between algebraic spaces and abstract sets. Cantor actually says that his concept of Mächtigkeit is ***different, yet similar*** to that of Steiner. So one can reasonably speculate that had mathematicians studied Cantor more closely, they might have discovered the theory of categories fifty years before Eilenberg and Mac Lane. But the 'foundational' culture has somehow prevented even those foundationalists who know something of Steiner from noticing the significance of this connection.

Let us now turn to a more precise description of '*Mengen*' and '*Kardinalen*'. A '*Menge*' has an underlying ensemble of points, but, more importantly, it is both variable and *cohesive*, features not possessed by an abstract set except in a degenerate way. Today, one habitually uses the term 'topology', to indicate, for example, a cohesive structure possessed by a line (on which might be defined, for instance, a function with a Fourier expansion). Now of course the line possesses a topology in the usual sense, but also present are many other similar structures better adapted to particular problems. Not only must one study these particular technically defined structures, one must also have a *general* conception of spaces with cohesion. It seems to me that, in his mathematical journey from number theory to Fourier analysis to '*Mengen*' to '*Kardinalen*', Cantor probably also thought something similar to this:

Cohesion of a topological nature we may regard as an *objective* cohesion; on the other hand, there is also cohesion of a *subjective* sort, arising from my *coming to know* the points of a '*Menge*' in a certain way—for example, as the values of a particular recursive function. That is, in my subjectivity, through the growth of my knowledge of a certain '*Menge*', I may come to know a certain point before I come to know another one. The succession of the appearances of the points does not usually have an objective mathematical significance, but nonetheless may be of relevance in certain contexts, for example, those involving calculation. This too is a type of cohesion which we may call *subjective*. (A good example of a type of set possessing subjective cohesion is that of recursive set, which is 'traced' by various threads generated by particular recursive functions.) It seems that both of these types of cohesion were recognized by Cantor: this is why he used a double bar to indicate the double abstraction which may in general be involved in passing from a cohesive *Menge* to its associated *Kardinale*.

Let us suppose that M is a particular category of *Mengen*, for example, one of the various categories of topological spaces, the category of recursive spaces, one of the various categories of combinatorial spaces, etc. Cantor says that we can take the 'pure' set of points of any such space, thus arriving at a cardinal. A contemporary illustration of this process is provided by a color television picture with its subtle contrast of color and detail furnished by advanced technology. We can turn down the color knob and turn up the contrast knob until nothing remains but stark white dots on a black background with even, we may imagine, the outlines of figures suppressed. The picture with all its beautiful colors is a '*Menge*': but in order to study effectively a certain superficial (but necessary) aspect of the picture we may be compelled to consider the 'bag' of dots or points obtained in the way we have just described. In carrying out this process of abstraction one forgets temporarily all beautiful particular features, in order to concentrate just on the points, now deprived of qualities, yet still equinumerous with those of the colored picture. The result still seems also to be a *Menge*, but a *Menge* of a degenerate sort. Thus, we find that every cardinal gives rise to a *Menge* of a type which we shall call discrete and that in fact we have a pair of adjoint functors:

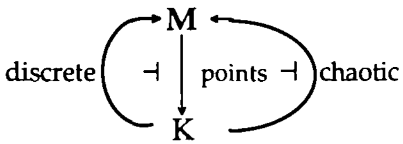


Thus a map from a cardinal K to the points of a *Menge* M is a map of abstract sets, that is, a map of a completely general kind with no condition of 'continuity' or preservation of cohesion: however, specifying such a map is equivalent to specifying a continuous map from discrete(K) to M. The continuous maps in the other direction are by no means arbitrary for most M: for example, there will be no nonconstant **M**-maps $M \longrightarrow$ discrete(2) if M is connected. In fact, for many **M** the foregoing clause serves well as a definition of which objects M of **M** are to be regarded as connected,[1] for the maps of **M** must preserve cohesion, and M may have a great deal of cohesion, while of course 2 has none. Thus a nonconstant map of the kind indicated is only possible if there is a break in the cohesion of M. A nonconstant map *to* a discrete space, if we can find one, would be the surest way of ***distinguishing*** two points of a given space. As is always the case with adjoints, if we take the case in which K is points(M) and the map the identity in K, we obtain a canonical map in **M** from discrete(points(M))

[1] This definition of connectedness is correct not only because it has useful technical consequences, but more fundamentally, because it corresponds to an objective concept: a '*Menge*' possesses cohesion *in general*, but might in particular have two parts, each cohesive in itself, but with no cohesion 'between' them. In this situation there would exist an **M**-map preserving all the cohesion there is to preserve, but mapping the two parts to the respective points of the space discrete(2) whose cohesion is nil.

to M itself which may be regarded as the best possible approximation to M 'from the left' given only its cardinality.

Now the 'points' functor also has a *right* adjoint, which is sometimes called the *codiscrete* or *chaotic* functor. The chaotic and discrete spaces determined by a cardinal are often completely different, so that, for example, all **M**-maps chaotic(K_1) $\longrightarrow$ discrete(K_2) are *constant*, whereas the **M**-maps in the opposite direction are (for two reasons) in exact correspondence with the arbitrary **K**-maps $K_2 \longrightarrow K_1$.



$$\frac{M \longrightarrow \text{chaotic}(K)}{\text{points}(M) \longrightarrow K}$$

Figure 1

Here the horizontal bar is an abbreviation to indicate that there is a natural bijection between the **M**-maps with domain and codomain as indicated above the bar and the **K**-maps indicated below the bar.

In general, a discrete space is completely deficient in its cohesion so that each point remains forever itself and no motion is possible, *i.e.*, no map from a connected space to it can pass through two distinct points. By contrast, a chaotic space is so excessive in its cohesion that any point can be moved to any other point without any 'effort', that is, with no attention paid to the nature of the space-'time' which might be used to parameterize the motion. Thus we may say that points in a discrete space are *distinct*, while points in a chaotic space are *indistinguishable* if chaotic spaces are connected.

Interesting categories of *Mengen*—combinatorial or bornological in nature—can be found which are, in a certain sense,[2] generated by the chaotic objects only, despite the fact that they contain objects with arbitrarily complicated higher connectivity properties. However, these are extreme special cases and I want to continue for a while to discuss the more general situation in which we are given an arbitrary category **M** of *Mengen*, itself containing two opposed subcategories of discrete and codiscrete objects, each essentially identical with a category **K** of *Kardinalen* (see Figure 2). The two composite functors are both isomorphic to the identity of **K**:

$$\text{points} \circ \text{discrete} = 1_K = \text{points} \circ \text{chaotic}.$$

That is, if to the canonical map discrete(points(M)) $\longrightarrow M$ we apply the

[2] That is, for such special **M**, a knowledge of all the special maps chaotic(K) $\longrightarrow M$ suffices to determine the arbitrary object M completely. This phenomenon has been studied by topologists for over 50 years under the name 'simplicial complexes'.

functor 'points', we obtain an isomorphism of cardinals

$$\text{points}(\text{discrete}(\text{points}(M))) \xrightarrow{\sim} \text{points}(M).$$

Similarly, using the canonical map $M \longrightarrow \text{chaotic}(\text{points}(M))$, applying the same functor 'points' as before yields an isomorphism of cardinals

$$\text{points}(M) \xrightarrow{\sim} \text{points}(\text{chaotic}(\text{points}(M))),$$

even though the two original canonical maps themselves are usually very far from being isomorphisms of *Mengen*. Thus the contradiction objectified in this system of adjoint functors explains the 'inconsistency' which

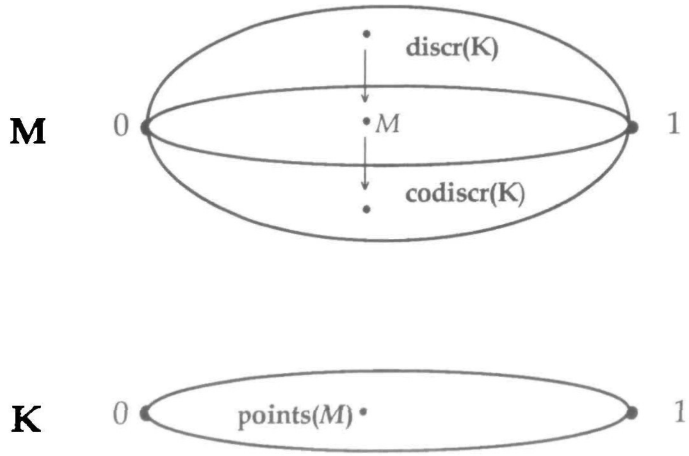


Figure 2

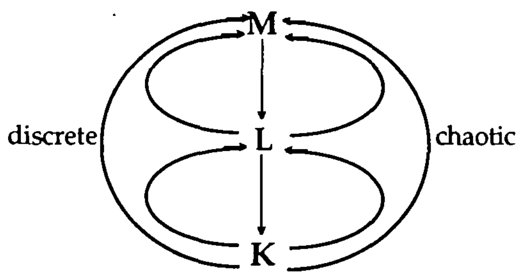
blocked Zermelo's understanding. For the cardinal points(M) associated with M is, as we have just seen, isomorphic both to the cardinal associated with the space discrete(points(M)) and to that associated with the space chaotic(points(M)): yet in the former all points are distinct, and in the latter, indistinguishable.

The 'inconsistency' of diversity versus indistinguishability, of having a definite number of points, and yet these points being indistinguishable by any property, seemed to Zermelo an irresolvable contradiction. The explicit use of adjoint functors between categories finally enables its truly productive nature to be revealed for all to see.[3]

[3] There are also toposes in which there are few connected objects and in which discrete(K) = codiscrete(K) for all K less than a measurable cardinal, and yet 'codiscrete $\neq$ discrete' is the main feature in the sense that maps from codiscretes determine all objects: for example, the topos of bornological sets (in which linear algebra becomes functional analysis).

Let us return to the configuration depicted in Figure 1 . It is natural to describe this configuration as an *adjoint cylinder*. Not only are the three functors involved adjoint, in the sense explained, but moreover the two composites at **K** are the identity. Such adjoint cylinders I propose as the mathematical models for many instances of the *Unity and Identity of Opposites*: this may equivalently be regarded as a property of the single functor 'points', in view of the essential uniqueness of adjoints. That is, **M** unites the discrete and codiscrete; **K** is a sub'object' of **M** in two opposite ways. Recall that the term 'subobject' always implies an inclusion; if we ignore the inclusion, we obtain **K** identically. The opposition here is expressed precisely by the two opposite senses of adjointness; it does *not* mean that the two sub'objects' are disjoint, since here they overlap in the two 'truth values' 0, 1 (see Figure 2).

One can take this notion of Unity and Identity of Opposites further. For after having grasped the basic features of *Kardinalen*, we can start to re-examine *Mengen*, *i.e.*, return to mathematics, equipped with more precise tools.[4] One now finds that there are often further adjoint cylinders in between those depicted in Figure 1:



Here the intermediate category **L** is less abstract than **K** and contains more information about the 'real' objects in **M**. On the other hand, **L** is simpler than **M** and perhaps more amenable to computation.

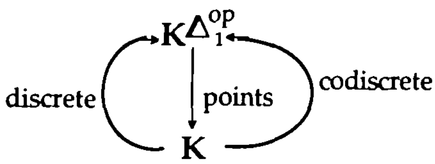
As an example of such an intermediate category, start with the three element monoid

$$\Delta_1 = \{1, \partial_0, \partial_1\} \qquad\qquad \partial_i\partial_j = \partial_i \qquad i,j = 0,1$$

and consider the category $\mathbf{K}^{\Delta_1^{op}}$ of right Δ_1-sets. The objects of this category are also known as *reflexive directed graphs*. We then have an adjoint

[4] Perhaps Cantor himself intended to do this, since at one point he suggests that the difference between ponderable matter and ethereal matter might just be a question of cardinality: a suggestion, which although probably inadequate, does at least show an intention that this abstract machinery should be applied to a pressing problem.

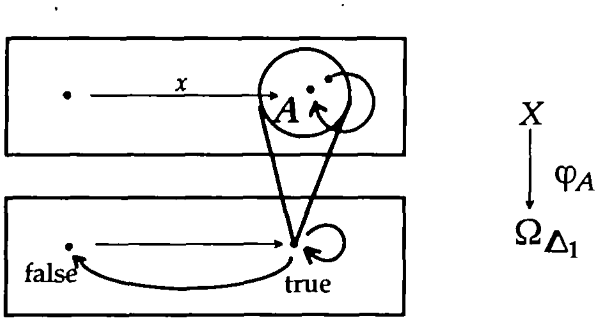
cylinder:



Here the points functor omits all the connecting arrows from a given graph and adjointness forces the codiscrete graph on a given set of points to have exactly one arrow connecting each ordered pair of points. The points functor is actually representable by the graph possessing a single point and only the degenerate arrow at that point, which can also be seen as a special case of the left adjointness of the discrete graph. Given an interesting graph X containing some information, any map of its points into the points of the associated codiscrete graph can be uniquely extended to a graph morphism from X itself because each arrow in X has exactly one place to go in the codiscrete graph.

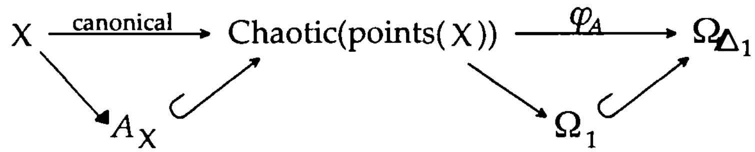
This example is worth a little more discussion since both $\mathbf{K}$ and $\mathbf{K}^{\Delta_1^{op}}$ are *toposes*. $\mathbf{K}$ forms a Boolean topos with truth value object $\Omega_{\mathbf{K}} = 2 = 1+1$. It is important to note that the fact that $\mathbf{K}$ is Boolean does not necessarily imply that its objects are totally abstract sets, since when $\mathbf{M}$ is (the category of *Mengen* associated with) algebraic geometry over a non-algebraically-closed field, the natural base topos $\mathbf{K}$ is a Boolean topos involving actions of the Galois group. (This summarizes part of Galois' great achievement.)

By contrast, the truth value object Ω_{Δ_1} in $\mathbf{K}^{\Delta_1^{op}}$ turns out to be a five-element graph which cannot be disconnected as a sum in any non-trivial way.



Given any subgraph A of any graph X, the arrows of X are partitioned into five kinds relative to A: those which are totally or truly in A, those which are completely outside A, those which enter A (as does x in the figure), those which leave A, and those which make an excursion from A only to return. The truth value object is in some sense the pure case in which all these possibilities are realized by taking A to be the single point 'true'. But on the other hand, for an arbitrary configuration A in an arbitrary X there is a unique graph morphism ϕ_A to Ω_{Δ_1} which maps A to 'true': the *characteristic morphism* of A.

We can use the elements of Ω_{Δ_1}—the ***truth values***—to start measuring things. Consider for example the canonical map from any graph X to the codiscrete graph on the points of X; its image is a subgraph A_X of the chaotic graph which therefore has a characteristic morphism to Ω_{Δ_1}.



Since this particular subobject contains all points, the characteristic morphism actually factors through the subobject Ω_1 consisting only of the loop at 'true' and 'true' itself. Thus an arrow of the chaotic graph is really just any pair of points but it goes to 'true' iff there exists an arrow in X between those points; otherwise it goes to the loop. So we have a 'measure' of the presence of arrows in X.

Can we find further adjoint cylinders between abstract sets and graphs? No, because graphs are one-dimensional. However, the graphs themselves may well occur 'sandwiched' between abstract sets and some more complex category **M**. The crucial condition for this to happen is the presence in **M** of an object T which has exactly two points and three endomaps in **M**, for then we can use T to initiate a deeper analysis of arbitrary objects M by defining a T-*edge* of M to be any map $T \longrightarrow M$, or, in other words, a point of the function space M^T. In this way we define a functor from **M** to $\mathbf{K}^{\Delta_1^{op}}$ which in many cases will possess both left and right adjoints, giving rise to an adjoint cylinder, and thus decomposing the cylinder structure of *Mengen* over *Kardinalen* into a stack of *two* adjoint cylinders. As a result, we can obtain upper and lower 'bounds' on a *Menge* once we know its graph.

The method suggested by Cantor which we are trying to make explicit here is roughly this: Among the many categories of cohesion and variation rich with mathematical interest, let us sharpen in particular our knowledge of the discrete and constant whose interaction with the rest serves as one tool for studying them all. It can be used in conjunction with another important tool as follows: If a certain small category $\mathbf{A} \subset \mathbf{M}$ has been somewhat understood, then maps $T \longrightarrow M$ for T in $\mathbf{A}$, but M less known can be reasonably deemed 'figures in M of shape T', and

$$\text{points}(M^T)$$

is the cardinal of such figures, depending functorially on M, with the contravariant functorality in T giving the 'structure' of a right $\mathbf{A}$-set expressing incidence relations. Often the three adjoints emphasized above are joined by a fourth, components $\dashv$ discrete $\dashv$ points $\dashv$ codiscrete , assigning to each *Menge* M the cardinal representing its maps into discrete *Mengen*. But then the 'higher connectivities' of M are represented by components(M^T) for suitable T such as tori and spheres. For a category $\mathbf{M}$ in which variation rather than cohesion is most important, the 'same' four adjoints relating it to $\mathbf{K}$ are usually called rather

$$\text{orbits} \dashv \text{stationary} \dashv \text{equilibria} \dashv \text{chaotic},$$

where the last only sometimes exists.

The dialectic reflected in this method is that between imagined Being in general with all its interlocking categories of cohesion and variation on the one hand and the extreme special case of discreteness and constancy on the other. Proper understanding of this dialectic may put into a different light debates over 'foundational' questions such as whether there is an infinite set of reals of cardinality less than the continuum, or whether there is a large measurable cardinal. The answers are clearly *no* if we are pursuing constancy toward an extreme, because Gödel's L construction would otherwise give us something still more constant. (It might seem that Gödel's L would rest, if anything does, on von Neumann's *a priori* ε-chains, but that was refuted twenty years ago by William Mitchell who showed how to construct it from a suitable given *category*, structured only by composition of maps; however, to my knowledge this has not been pursued since.) On the other hand, the answer is clearly *yes*, the GCH is false, if we are considering almost any cohesive variable topos of independent mathematical interest. Work of the past thirty years, starting with Cohen, showed that the GCH and many similar constancy properties can actually fail even in toposes where the variation is sufficiently hidden to escape detection by the more obvious constancy indicators such as Booleanness and the axiom of choice. There is a principle, elucidated over the past fifteen years, to the effect that if a statement in internal logic can be refuted by cohesion and variation, then it can also be refuted by variation alone; thus to deal with cohesion itself, as in algebraic topology, requires going beyond the internal logic.

Cantor's theory of *lauter Einsen* may be compared with Galois' contribution to algebraic geometry: The study of spaces defined by equations and quantities varying over them had been much developed before him, but Galois' incisive work on the case of zero-dimensional algebraic spaces became indelibly intertwined with the subsequent continuing study of higher-dimensional ones. Similarly, the very special Boolean toposes continue to come up from both sides in the analysis of general toposes, further unfolding the results of the method which Cantor envisioned.

Acknowledgements

This article would not have come about without Michael Wright's devotion to science which led to his unique sponsorship of the meeting held at Old Cavendish Laboratory in Cambridge in summer 1989. That meeting was presided over by host and chairman Michael Redhead. I thank these and the other organizers and participants for the successful collaboration whose further developments are still under way.

Special thanks go to John Bell for creatively preparing an earlier draft of this paper, which is a segment extracted from my series of lectures at that meeting.

ABSTRACT. For 20th century mathematicians, the role of Cantor's sets has been that of the ideally featureless canvases on which all needed algebraic and geometrical structures can be painted. (Certain passages in Cantor's writings refer to this role.) Clearly, the resulting contradiction, 'the points of such sets are distinct yet indistinguishable', should not lead to inconsistency. Indeed, the productive nature of this dialectic is made explicit by a method fruitful in other parts of mathematics (see 'Adjointness in Foundations', *Dialectica* 1969). This role of Cantor's theory is compared with the role of Galois theory in algebraic geometry.