Display of graphics and their applications, as exemplified by 2-categories and the Hegelian "taco"

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A graphic monoid M satisfies identically xyx = xy and an application of M is a right M-set. Every left ideal of such an M is also a right ideal, simplifying and structuring the study of the topos of applications. An informal process of displaying pictures of graphics and applications is exemplified, with conjectured use in the organization of knowledge. The Hegelian organization of knowledge is concretely realized in terms of adjoint functors on "any" mathematical category, and is used to give a precise definition of the dimension needed for a display. A central fragment of the Hegelian scheme is revealed as an 8-element graphic, whose suggestive display has reminded some of a taco.

I. INTRODUCTION

By a graphic we will mean any finite category each of whose endomorphism monoids satisfies the identity xyx = xy ; in particular, a graphic monoid is a graphic category with one object. By an application of a graphic category we will mean any right action of it on finite sets (i.e. any contravariant finite-set-valued functor on it). If I is any object of a graphic G , then G(-,I) is a particular application (often called the right regular representation in the case of a monoid) and together these give a full embedding of G into the topos of all applications of G , to which we freely apply the Cayley-Dedekind-Grothendieck-Yoneda lemma. If X is any application of the graphic G , then the "comma" category G/X (whose objects are the elements of X and whose morphisms determine the action via the discrete fibration property of the labelling functor

$G/X \longrightarrow G$) is again a graphic. Thus each particular application X of G provides one way $G' \longrightarrow G$ of expanding the graphic G into a more detailed graphic G'. Even though graphic monoids G play a central role, we must also deal with graphics such as G/X with many objects. Similarly, the category $\bar{G}$ of all retracts of objects of G (which may be constructed either abstractly to have as objects the idempotents of G or concretely as a full subcategory of the category of applications of G; note that in the former guise it is "a itself" which plays the role of 1_a) will again have many objects - indeed the graphic identity $xyx = xy$ implies $x^2 = x$ so that if G is a monoid then $\bar{G}$ has an object for every element of G (some of those objects may be isomorphic in $\bar{G}$). The interest of $G \hookrightarrow \bar{G}$ is that it induces an equivalence between the associated toposes of applications. We intend to associate with each graphic (by a compelling though not yet well-defined process) a "display" which will reveal much of its structure. We do associate a well-defined distributive lattice which is itself a standard application and which may be considered to consist of refined "dimensions" in that it parameterizes all the ranks in a Hegelian analysis of the topos of all applications; through this distributive lattice there is a well-defined ascending sequence, obtained by the Hegelian process of "resolution of one unity of opposites by the next"; the length of this sequence is the geometrical dimension of the display in our numerous examples.

What is especially striking is that the Hegelian analysis of <u>any</u> topos turns out to involve <u>graphic</u> monoids which are in fact bicategories. Thus, the organization of any branch of knowledge, insofar as it can be mathematical (i.e. teachable), may in some measure reflect itself in graphic displays. Though proposed [0] nearly 200 years ago, the Hegelian method of analysis has been widely under-utilized since then; "conflicting" ideological claims either that it is inconsistent or that it is too wonderfully fluid to be made mathematical have conspired to prevent its being widely taught. We believe that we have through modest examples shown it to be consistent (and non-trivial) and that much of the method should be made mathematical, which would help those who seriously want to use it, even that part which remains fluid.

By a <u>constant</u> c in a graphic monoid is meant an element such that cx = c for all x . The three element monoid with two constants ∂_0, ∂_1 (so $\partial_i \partial_j = \partial_i$) has as its applications all the <u>reflexive directed graphs</u>; that example plays a central role in [1,2] and suggested the name. Toposes of applications of such "constant" graphics with more than two constants were investigated in [2] , partly as a vehicle for explaining some basic topos theory and partly to determine how they were different from the two-constant cases in which $x\partial_0, x\partial_1$ denote the beginning and ending points of an arbitrary directed edge x. In the course of that work, the identity xyx = xy was discovered as the least common generalization of constant (x = c) and identity (x = 1); later I learned that it had been briefly mentioned as a purely formal generalization in [3] , where the <u>finiteness</u> was noted, and that in [4] a partial structure theorem for such monoids was proved as well as a structure theorem for certain more general monoids using these as one of the ingredients. (As for finiteness, it is immediate that the free graphic monoid on a finite set of letters consists of all words <u>without</u> repetitions, of which there are only $n! \sum_{i=0}^{n} \frac{1}{i!}$). So far I have not found any previous discussion of applications (in either sense).

In this paragraph (and the next) we make some imprecise remarks about possible uses. Retrieving stored knowledge presupposes some consciousness of the structure it has; this structure is in its particularity fixed by the storage process itself (and in its generality is partly a reflection of the content, i.e. of the nature of the knowledge stored). Thus in both retrieval and storage one needs to be explicitly aware of the kind of structure involved. Here we are momentarily accenting the "passive" aspect of the structure, the kind of structure that both codomain and domain of more "active" operations such as rewrite must have ("peeking" may be definable). Now it is commonly recognized that commutative operations such as Boolean intersection are involved, but also "something further". We here speculate that non-commuting systems of idempotent operations may capture some of the further subtlety. The arrangement of shelves in any science library shows that topological algebra $\neq$ algebraic topology and chemical physics $\neq$ physical chemistry, although these are in some sense "intersections". A feature which seems to be present is that a sub-branch b is not only a subset but reflects things x (not necessarily in b) to a part bx of b which is most relevant to x (bx is a single element in the generic case of $G(-,I)$ but the idea retains force in general applications).

As another example, we could assign to every page of every book the title page of the book that it is in; clearly this operation specifies the set of all title pages, but much more. Such idempotent operations need not commute but on the other hand would have a rather strong commutation relation reflecting the hierarchical structure of empty documents within folders within disks.... We have pursued the investigations summarized here in the hope that the "graphical" identity may capture many instances of this commutation relation. This hope was strengthened by the recent discovery that that identity arises in the Hegelian scheme of knowledge. It is said that the German philosopher Hegel, building on the work of Aristotle and in opposition to the eclectic listing of categories of sciences by his "metaphysical" predecessor Wolfe, proposed to generate the main categories by a single dialectical process. The great mathematician Grassmann, partly inspired by Leiniz, also emphasized the dialectical method in building up his geometrical theory of extensive quantities. What striking contrast between these, who advanced both knowledge and its organization, and those to whom $x \in x$ is a big issue and who lead us astray with library-catalogue paradoxes, when more conscious access to libraries is what is needed! [2)]

II. Elementary Consequences of the Basic Identity, with special reference to ideals

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We begin our calculations by pointing out some remarkable consequences of the graphic identity

$$aba = ab.$$

For any right action X of any monoid M, there is for any element x the <u>stabilizer</u>

$$\mathrm{Stab}(x) = \{a \in M \mid xa = x\}$$

<u>PROPOSITION 1</u> If M is a graphic monoid, then the stabilizer of any element x of any application X is a <u>saturated</u> submonoid: $ab \in \mathrm{Stab}(x) \Longrightarrow a, b \in \mathrm{Stab}(x)$.

Proof: $xab = x \Longrightarrow xa = xaba = xab = x$ and $xb = xabb = xab = x$.

For any action the part fixed by all M is a (trivial) subaction, but the part fixed by a single $a \in M$, which for idempotent a satisfies

$$Xa = \{x \in X \mid xa = a\} ,$$

is usually only a sub<u>set</u> (it <u>is</u> a functor of X).

<u>PROPOSITION 2</u> If M is graphic and $a \in M$ and if X is any application of M, then Xa is actually a sub-application, i.e. $x \in Xa \Longrightarrow xb \in Xa$ for all $b \in M$.

Proof: $xa = x \Longrightarrow (xb)a = xaba = xab = xb \Longrightarrow xb \in Xa$

One of the most powerful consequences of the graphic identity is that

every left ideal is a right ideal

which follows from the next proposition, using the fact that every ideal of either kind is a union of principal ideals.

PROPOSITION 3 For any element of any graphic monoid M

$$aM \subseteq Ma$$

Proof: For every x there is an element x^a for which

$$ax = x^a a,$$

namely, we can take $x^a = ax$.

Since every element of a graphic monoid is idempotent, it follows trivially that

every left ideal S is idempotent

in the sense that $SS = S$. For a general monoid, this would be equivalent to "for every a, there are u,v for which $a = uava$". This would include all groups, and also the monoid of all endomaps of a 2-element set, which figures in [2]. Perhaps much of what follows could be generalized to all monoids satisfying the two boxed axioms above, but if we assume idempotence of elements, it can be shown that $aM \subseteq Ma$ implies the graphic identity.

Often Ma is much bigger than aM, but as a right ideal it is a finite union $\bigcup b_i M$ of principal right ideals. The smallest number #(a) of b_i required could be considered as a crude measure of the size of a.

PROPOSITION 4 $Ma = \bigcup b_i M$ iff

1) $b_i = b_i a$ for all i

2) for all x, $xa = b_i x$ for some i.

In particular, one of the b_i must be a itself.

Proof: $xa = b_i y$ for some y so $xa = b_i xa$ by idempotence. Thus $xa = b_i axa$ by 1) so $xa = b_i ax = b_i x$. Taking $x = 1$ proves the last remark.

Normally a principal ideal can have more than one generator, but in a graphic the elements are faithfully represented by right ideals:

PROPOSITION 5 In a graphic monoid, $aM = bM \Longrightarrow a = b$.

Proof: We have $a=bx$ and $b = ay$, hence by idempotence $a = ba$ and $b = ab$. But $a = ba = bab = bb = b$.

For principal left ideals we do not have faithfulness but we do have, since $Ma = Mb$ iff $a = ab$:

PROPOSITION 6 In a graphic monoid, $Ma = Mb$ iff $a = ab$ and $b = ba$ iff $\mathrm{Stab}(a) = \mathrm{Stab}(b)$ iff a,b are the images of ∂_0, ∂_1 under a homomorphism from the three element monoid with 2 constants.

Note that $aM \cap bM$, while a right ideal, is not usually a principal right ideal and is often even empty. But for principal left ideal this situation is simpler:

PROPOSITION 7
$$Mab = Ma \cap Mb$$
$$M1 = M$$

for any graphic monoid. Hence $Mab = Mba$.

Proof: $Mab \subseteq Mb$ is clear. By the graphic identity, we also have $Mab \subseteq Ma$. If an element x is in both Ma and Mb, then $x = xa$ and $x = xb$ by idempotence, so $x = xb = xab \in Mab$.

As Kimura [4] proved and used, the image CM of the homomorphism $M \longrightarrow$ (left ideals, $\cap$) thus defined is actually the universal homomorphism to any commutative graphic monoid (=semilattice). Schanuel (unpublished) showed, as suggested by Propositions 1 and 6, that this semi-lattice reflection CM can alternatively be constructed as part of the set of all saturated submonoids under the join operation on such (note that $Ma \subseteq Mb$ iff $Stab(a) \supseteq Stab(b)$).

Now we recall that in the topos of all applications of M, the truth-value application Ω is the one consisting of all right ideals of M, under the action of each $b \in M$ defined at A by

$$A:b = \{x \in M \mid bx \in A\}$$

which is easily seen to be another right ideal if A was. The universal use of Ω is: if $Y \subset X$ is any sub-application, then $X \xrightarrow{\varphi} \Omega$ defined by

$$\varphi x = \{a \in M \mid xa \in Y\}$$

is an M-equivariant morphism of applications, and the unique one for which

$$x \in Y \iff \varphi x = \text{true}$$

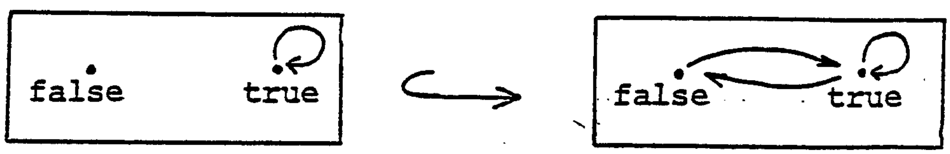
(where true $= M \in \Omega$) holds for all x in X. In general φx is thought of as the truth-value of the statement "$x \in Y$", which value just consists of all available acts which bring about actual truth. For example, in the case where applications = directed graphs, there are five truth-values, two of which are points, one is a loop at true, and the other two are edges connecting (in the two directions) true with false $= \emptyset$.

In the case of a graphic monoid we have shown (Proposition 3) that every left ideal is a right ideal. Even more remarkably, if we consider the sublattice $\Omega_{left} \subset \Omega$ (of the distributive lattice of all right ideals) which consists of the left ideals, we have

PROPOSITION 8 For a graphic monoid M, $\Omega_\ell \subset \Omega$ is a sub-application.

Proof: If S is a left ideal and $a \in M$, then $S{:}a = \{b \mid ab \in S\}$. We must show that this is again a left ideal. So suppose $ab \in S$ and that $c \in M$; we must show $cb \in S{:}a$, that is that $acb \in S$. But $acb = (aca)b = acab \subseteq Mab \subseteq S$ since S itself was a left ideal.

Even though the inclusion of posets $\Omega_\ell \subset \Omega$ has both a left adjoint $(A \longmapsto MA)$ and a right adjoint, neither of the latter is a morphism of applications. For example, for directed graphs (where $\circlearrowleft <$ true in the ordering, which we suppress) the inclusion in question is



which admits no graph-theoretic retraction (order-preserving or not). Note that $aM \subseteq bM \Longrightarrow Ma \subseteq Mb$.

Although applications in general do not have left actions, we can ask: For which inclusions $Y \subset X$ of applications does the corresponding characteristic map $\varphi : X \longrightarrow \Omega$ actually factor through the sublattice $\Omega_\ell \subset \Omega$ of left ideals? In the example of directed graphs, the above picture shows the answer to be: those subgraphs Y of the graph X for which no directed edge of X enters Y or leaves Y except on excursion, i.e. $x\partial_0 \in Y \Longleftrightarrow x\partial_1 \in Y$ for all x.

Now in the generic application $X = M$, the left multiplication by a may be considered as the reflection of an arbitrary x to (the "most relevant element of"?) the fixed point set Xa In a particular application X, left multiplication by a is usually not defined. However, by proposition 2, $Xa \subset X$ is a sub-application, and hence by the universal property of Ω there is a unique characteristic map $\varphi_a : X \longrightarrow \Omega$, and we have $aM \subseteq \varphi_a x$ for all x, for even $Ma \subseteq \varphi_a x$. We may ask, when is $\varphi_a(x) \varepsilon \Omega_\ell$? By definition

PROPOSITION 9 $\varphi_a(x) \varepsilon \Omega_\ell$ iff

$$\forall b, \lambda \varepsilon M \ [xba = xb \Longrightarrow x\lambda ba = x\lambda b]$$

PROPOSITION 10 If $X = M$ and if M consists only of constants and 1, then $\varphi_a x \varepsilon \Omega_\ell$ for all $x, a \varepsilon M$.

Throughout this paper we consider only the category of right actions or "applications" (categories of left actions are treated very briefly in the examples in [2] and have rather different properties). Thus it must constantly be kept in mind that whenever we attribute a property such as "connectedness" to a left ideal S, we are using our proposition 3 to consider S as an object in the category of (right) applications-connectedness of S as a left action would mean something quite different! Similarly, when the set Ω_ℓ of left ideals is considered as an object in a category, it will be (either as a lattice or) according to proposition 8 as an application.

III. Elementary Examples and their Intuitive Displays

In preparation for listing some examples of graphics, let us make explicit some facts about the role of constants.

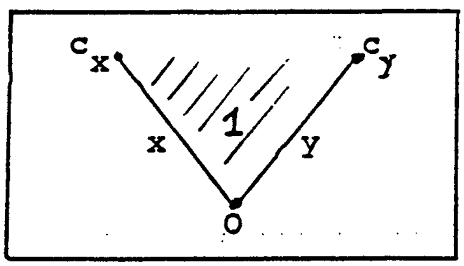
PROPOSITION 11 Every graphic monoid contains constants.

Proof: Since we have assumed finiteness, let c be the product, in some chosen order, of all the elements of the monoid. Then $cx = c$ for any x, since x already occurs first as a factor of c, and the basic identity cancels second occurrences.

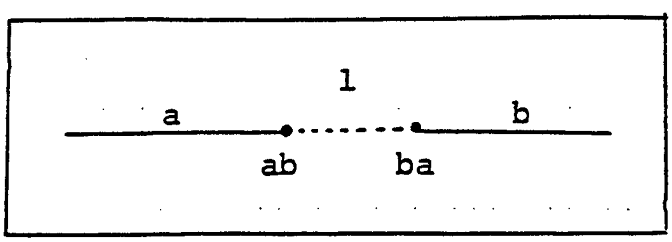
For example, the free graphic monoid on n generators has $n!$ constants, since all words of maximal length are distinct. On the other hand, all those can be collapsed to one without imposing any further relations between words of shorter length. Thus (not only commutative) examples may have a unique constant.

<u>PROPOSITION 12</u> If c is a constant, then so is ac for any a. Thus Ma includes all constants, hence any non-empty left ideal contains all constants. Also if there is a unique constant o, we have $ao = o$ for all a.

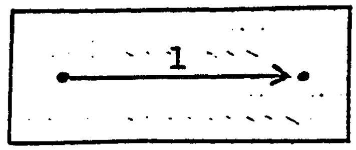
The left action of M on the set T_o of all constants of M may thus fail to be faithful. However, we can always adjoin new constants, for example via the sub-representation $M \cup X$ of the faithful <u>left</u> regular representation of M on $X = M$. If we do that to the four-element free semilattice on two generators x, y, we get a six-element graphic whose display will turn out to be the two-dimensional picture



Of course any <u>free</u> graphic monoid does act faithfully (on the left) on its constants. For example the five-element free graphic monoid on two generators a,b has the two constants ab and ba, on which the generators act by interchanging them; however, its display will turn out to be the one-dimensional:



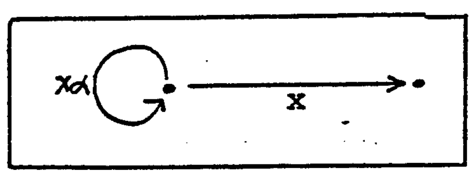
The graphic monoid $\mathbb{A}_1$ with only three elements, two of which are constant, is displayed



and all its applications are "one-dimensional", being directed graphs. It is of wide use in analyzing more complicated graphics, for example, consider the graphic monoid M which is freely generated by two elements α, ∂_1 subject to the one relation $\partial_1\alpha = \partial_1$ and define $\partial_o = \alpha\partial_1$. Then

$$\partial_o\partial_1 = \alpha\partial_1\partial_1 = \partial_o$$
$$\partial_1\partial_o = \partial_1\alpha\,\partial_1 = \partial_1\alpha = \partial_1$$

so that any M-application has in particular an underlying directed graph, but is more in that α also acts on the directed edges. In addition to the defining relation, we have $\partial_o\alpha = \alpha\partial_1\alpha = \alpha\partial_1 = \partial_o$ so that both ∂_i remain constants even in M. The definition of ∂_o says that any $x\alpha$ ends at the beginning of x, but moreover $\alpha\partial_o = \alpha^2\partial_1 = \partial_o$ so that $x\alpha$ is a <u>loop</u> at $x\partial_o$. Thus every edge x in an application carries with it a picture



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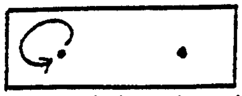
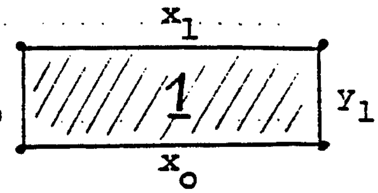
if x is interpreted as a process, we might consider $x\alpha$ as the "preparation" necessary for x. In order to represent M faithfully by endomaps, consider one more constant * together with $\partial_o\partial_1$ and define an operation on this three-element set by $\alpha(\partial_o) = \alpha(\partial_1) = \partial_o$, $\alpha(*) = *$. The left-ideal lattice Ω_ℓ has four elements

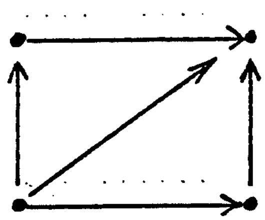
$$\emptyset \subset M\partial_o = M\partial_1 \subset M\alpha \subset M$$

but $M\alpha =$  is not "connected", which will mean that even as a graphic in its own right, M must be displayed as one-dimensional. This contrasts with

$\Delta_1 \times \Delta_1 =$ 

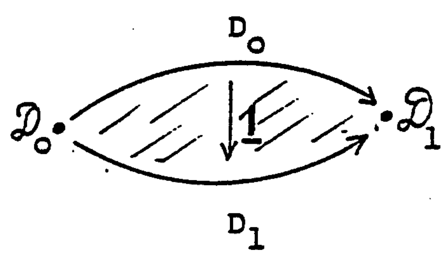
$x_i y_j = y_j x_i$

a two-dimensional, nine element graphical monoid, which like the above M also receives a homomorphism $\Delta_1 \longrightarrow \Delta_1 \times \Delta_1$, say the diagonal. Along the latter, we also get an underlying graph, whose display is



In general, if every homomorphism $\Delta_1 \longrightarrow M$ is assigned a color, then all the underlying graph structures of M could be simultaneously displayed.

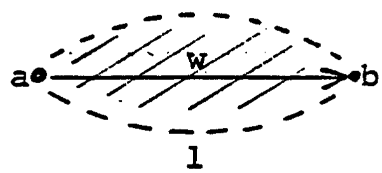
For another important example, recall that graphs underlie the theory of categories, but that there are also 2-categories; underlying the latter are <u>2-graphs</u>, the generic example of which is



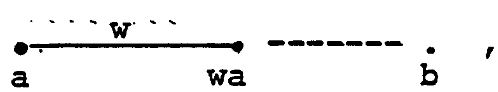
This can be made into a five element (four generator) graphical monoid by defining $\mathcal{D}_i \mathcal{D}_j = \mathcal{D}_i$, $D_i D_j = D_i$, $D_i \mathcal{D}_j = \mathcal{D}_j$, $\mathcal{D}_i D_j = \mathcal{D}_i$
Every 2-category (for example the 2-category of all graphics, all functors between these, and all natural transformations between those) has an underlying application of this monoid, in which

the $\mathcal{P} D_i$ are the domain and codomain "functors" of any "natural transformation" $\mathcal{P} = \mathcal{P} \cdot 1$ and FD_i are the domain and codomain "categories" of any "functor" F . The lattice Ω_ℓ turns out to be a <u>linearly</u>-ordered set isomorphic to $\{-\infty < 0 < 1 < 2\}$ where 0 stands for the constant $\mathcal{D}_i$ but 1 stands for the left ideal $\bullet \rightrightarrows \bullet$ which is already connected as a right ideal, hence (by the general theory to be described presently) the graphic itself has a <u>two</u>-dimensional display.

If to a nontrivial graphic monoid we adjoin a new identity element, so that the original monoid becomes a connected left ideal in the new monoid, we get again a graphic monoid of dimension at least two. If we do this to Δ_1 , and denote the original identity element by w , we see



that w is more of a "core" than a "boundary", and moreover that, since this is a homomorphic image of



dimension can be increased by homomorphic image. Since $w\partial_i = \partial_i$, in the underlying-graph display of M the cloud 1 condenses into another arrow parallel to w .

In order to describe a certain class of examples, two more propositions will be helpful.

<u>PROPOSITION 13</u> The lattice Ω_ℓ of left (=bi) ideals in a graphic monoid M is linearly ordered iff for every pair a,b of elements in M

$$a = ab \quad \text{or} \quad b = ba \quad .$$

Proof: This is the condition that $Ma \subseteq Mb$ or $Mb \subseteq Ma$, i.e. that the (semilattice) commutative reflection CM be linearly ordered. But the left ideals of CM are included surjectively into the left ideals of M , and the left ideals of a linear semilattice are clearly linearly ordered.

PROPOSITION 14 (Schanuel) Suppose that the endomorphism monoid of an object A in a category (such as $\overline{M}$) satisfies the graphic identity, and that B is any other object. Then there is at most one splittable epimorphism $A \xrightarrow{p} B$. In case A,B are retracts of a common graphical object I with idempotents a,b then p exists iff $Mb \subseteq Ma$, where M is the endomorphism monoid of I.

Proof: Suppose p has splitting section s, but that also q has splitting section i; that is $ps = 1_B = qi$. Then of course sp and iq are idempotents at A, but since A is graphic also ip and sq are idempotents. Better

$$sq = s(pi)q = (sp)(iq)(sp) = s(piqs)p = sp$$

so that q = p because s is a monomorphism. It is easily checked that at least one p exists iff b = ba, in the $\overline{M}$ case.

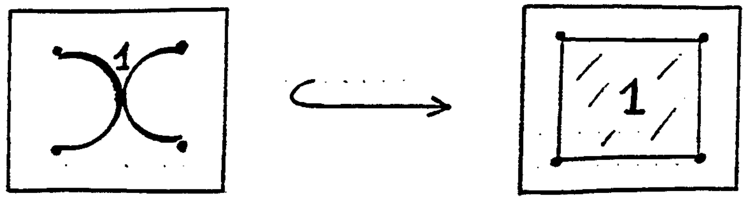
Thus in any graphic the subcategory of all splittable epimorphisms forms a poset. If

$$A = B_n \xrightarrow[p_n]{} B_{n-1} \to B_{n-2} \to \cdots \to B_0 \to B_{-\infty}$$

is any linear family of splittable epimorphisms in any category, and if we consider for each k any non-empty finite set of sections $B_{k-1} \underset{s}{\hookrightarrow} B_k$ for p_k, then the submonoid of endomorphisms of A obtained by considering all composites will be a graphical monoid. Special interest will attach in part IV. to the case where we consider two sections for each p_k.

Note that the unique retraction $I \to aM$ "represents" on the level of elements all the unique inclusions $X_a \hookrightarrow X$ in the topos of applications of M.

The (one-dimensional) graphic monoid with four constants and five elements (which was described as a "bare unity" in [2]) can be embedded in the two-dimensional $\Delta_1 \times \Delta_1$; the one dimensional connection might be displayed as



Another interesting embedding is

<u>PROPOSITION 15</u> The free graphic monoid F on two generators a,b can be embedded in $\Delta_1 \times \Delta_1 \times \Delta_1$.

Proof: Note that $M = \Delta_1 \times \Delta_1$ has a pair of elements f,s such that $s \neq fs = sf \neq f$. For any such M, F can be embedded in $M \times \Delta_1$ by sending $a = \langle f, \partial_0 \rangle$, $b = \langle s, \partial_1 \rangle$.

IV. Unity and Identity of Opposites in Bicategories and precise Definition of the refined and coarse Dimensions of Displays

In order to clarify the notion of dimension which arose in our intuitive displays of graphics, as well as to provide an infinite number of examples of graphics arising from non-idempotent mathematical structures, consider the following

<u>DEFINITION</u> A functor $\mathcal{A} \longrightarrow \mathcal{B}$ will be called a unity-and-identity-of-opposites (UIO) iff it has both left and right adjoints and one of the latter is full and faithful (hence both are). Then, denoting by L and R the two idempotent endofunctors of $\mathcal{A}$ obtained by composition, we have also $L \dashv R$ and $LR = L$, $RL = R$.

The two adjoints are the inclusions of two opposite subcategories united in $\mathcal{A}$, yet identical with $\mathcal{B}$. The terminal functor $\mathcal{A} \longrightarrow \mathbb{1}$ is a UIO iff $\mathcal{A}$ has both initial and terminal objects; the latter may be called non-being and pure being resp., and in general L is "non" whatever attribute (of $\mathcal{A}$) R is the "pure" form of. If $\mathcal{A}$ is a topos then $\mathcal{B}$ will automatically be a topos as well; this applies to our fundamental class of examples, where $\mathcal{A}$ is the category of all applications of a given graphic. In case $\mathcal{A}$ is a topos, R is called the $\mathcal{B}$-sheafification, and "non" sheaves may be called $\underline{\mathcal{B}\text{-skeletal}}$. The set of all UIO's with a given $\mathcal{A}$ forms a poset with respect to the "greater than" ordering

$$\begin{array}{ccc} & \mathcal{A} & \\ \swarrow & & \searrow \\ \mathcal{B}' & \dashrightarrow & \mathcal{B} \end{array}$$

This poset is often small even when $\mathcal{A}$ is large and is often a complete lattice, as is shown in a forthcoming joint paper with Kelly [5] . For example

PROPOSITION 16 If $\mathcal{A}$ is a category of all right actions (on sets) of a small category C , then the poset of UIO's with domain is equivalent to the poset of all idempotent two-sided ideals in the category C , with the empty ideal corresponding to $\mathcal{A} \longrightarrow \mathbb{1}$.

Corollary: For the category $\mathcal{A}$ of all applications of a given graphic monoid M the poset of all UIO's is parameterized by the poset of all left ideals of M . In more detail, if S is a left ideal of M , then an application X is an S-sheaf iff every morphism $S \longrightarrow X$ in $\mathcal{A}$ is of the form $s \mapsto x \cdot s$ for a unique element x of X , and on the other hand the S-skeleton $L_S(X) \subset X$ of any application X is given by

$$L_S X = \bigcup_{s \in S} Xs$$

i.e. all those elements of X that are fixed by some $s \in S$. Moreover, (since idempotence is automatic and quite unlike the general case) (not only the suprema but also) the infima in this finite (distributive!) lattice are computed as ordinary (unions and) intersections.

We will attribute refined dimension $\leq S$ to all applications X which satisfy the "negative determination" $L_S X \xrightarrow{\sim} X$.

In particular, $\emptyset$ will also be called of dimension-∞ and T_0 = the set of all constants of M determines the subtopos $\mathcal{B}_0$ of all "codiscrete" applications so that 0-dimensional means "discrete": We will assume that M has at least two constants, which implies that Ω is connected ($\pi_0\Omega = 1$) and that the "components" functor $\mathcal{A} \xrightarrow{\pi_0} \mathcal{B}_0$ (extra left adjoint to the discrete inclusion) preserves finite products [1,2] . To define coarse dimensions 1,2,... we will use the following

DEFINITION: If $S \subseteq T$ are left ideals, say that T resolves the opposites of S , in symbols

$$S \underset{\mathcal{A}}{<<} T$$

iff every S-skeletal application is a T-sheaf, i.e. iff $R_T L_S = L_S$ Because of the nice properties of intersection mentioned in the corollary to Proposition 16, there is for every S a smallest S' which resolves the opposites of S ; we may call S' the "Aufhebung" of S . Then the Aufhebung of pure being versus non-being is pure becoming versus non-becoming, i.e. codiscrete (chaotic) versus discrete, since if $\emptyset$ is to be a T-sheaf, then there can be no maps $T \longrightarrow \emptyset$, i.e. T must be non-empty, but by Proposition 12, T_0 (= the set of all constants of M) is the smallest non-empty left ideal; thus $(-\infty)' = 0$ as claimed. Since, intuitively, one-dimensional figures are the dimensionally-smallest ones which permit connecting all those points that can be connected, still more satisfying is

PROPOSITION 17 $0' = 1$. That is, $\pi_0 L_T = \pi_0$ iff $R_T L_0 = L_0$. Thus T_1 is characterized as the smallest left ideal of M which is connected as a (right) application of M .

Proof: Composite adjoints are adjoint composites. Or, if discrete applications D are to be T-sheaves, then every $T \longrightarrow D$ must come from an element of D ; but elements of D are constant (non-becoming), hence every $T \longrightarrow D$ must be constant (e.g. for D = 2), hence T must be connected.

Corollary: If $M \setminus \{1\}$ is not connected, then M is one-dimensional, whereas if $M \setminus \{1\}$ is connected and is the "Aufhebung" of some S which is in turn an Aufhebung..., then M is at least two-dimensional.

Here the dimension of M itself is defined in terms of the length of the sequence $T_{n+1} = T'_n$; experience [6] with other examples suggests that this length is the dimension for small dimensions and a simple function of it for higher dimensions.

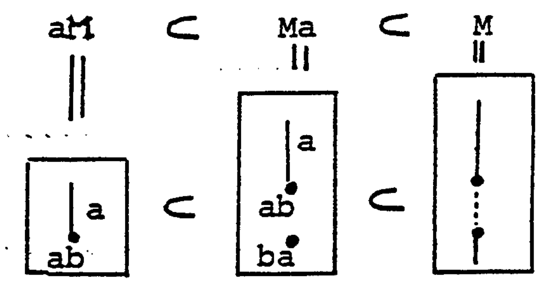
PROPOSITION 18 If M is the free graphic monoid on $k \geq 2$ generators, then dim $M = 1$.

Proof: Since "first letter of a word" is well-defined,

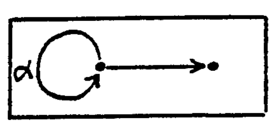
$$M \setminus \{1\} = \sum_{i=1}^{k} a_i M$$

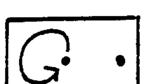
is a disjoint sum in the category of applications, hence not connected.

While principal right ideals are connected, principal left ideals need not be, for example, Ma in the free example on a,b :



An even smaller example of an "infinitesimal dimension" is provided by

$M =$ 

where  is a left ideal. But note that a left ideal which contains a connected left ideal is itself connected, for any t can be moved to a constant by the right action of a constant.

Now consider any category $\mathcal{A}$ with initial and terminal objects $\emptyset, 1$ and a double resolution of the latter by $\mathcal{C}, \mathcal{B}$ which successively climb $\mathcal{A}$-ward $\mathcal{A} \to \mathcal{B} \to \mathcal{C} \to \mathbf{1}$. Let r = pure $\mathcal{C}$, ℓ = non $\mathcal{C}$, R = pure $\mathcal{B}$, L = non $\mathcal{B}$. The first resolution means $r\emptyset = \emptyset$ (which implies $\mathcal{A}_{\neg\neg} \subset \mathcal{C}$ if $\mathcal{A}$ is a topos) while the second, $R\ell = \ell$ means that there are three (rather than four) subcategories of $\mathcal{A}$ "identical" with $\mathcal{C}$. Assume for simplicity that also $\ell 1 = 1$. Consider the category $\mathcal{M}$ of all endofunctors of $\mathcal{A}$ definable by composition from these and all natural transformations definable from the adjunction morphisms. $\mathcal{M}$ is a finite non-symmetric monoidal category, and there is only one object $q = Lr$ in $\mathcal{M}$ which does not have either a left or a right adjoint in $\mathcal{M}$ - it comes from the third embedding of $\mathcal{C}$ in $\mathcal{A}$.

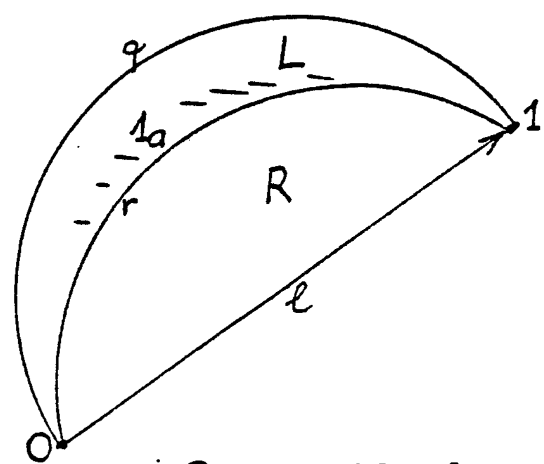
<u>PROPOSITION 19</u> The objects of $\mathcal{M}$ under composition constitute (up to equivalence) a graphic monoid of ($\leq$) eight elements which has five left ideals

$$\emptyset \subset \{0,1\} \subset [\ell, q, r] \subset [L, R] \subset [1_{\mathcal{A}}]$$

(where we have shown only the elements new at each stage). The middle of these (generated by any lower case letter) is already connected (by the right action of 0 !).

Thus the display of $\mathcal{M}$ is apparently

Hegelian "taco", a display of the 3-dimensional 8-element graphic monoid $\mathcal{M}$



which reminded some of a taco: All the meat of $\mathcal{A}$ is inside $1_{\mathcal{A}}$, while there are two identical faces L,R with a common edge ℓ and separate (but identical) edges q,r.

To finish the proof that $\mathcal{M}$ is really three-dimensional, we need only show that the Aufhebung of $[\ell,q,r]$ is just $[L,R]$, i.e. does not somehow jump all the way to the top $[1_\alpha]$ of the dimension lattice as happens in other examples. But $S = [\ell,q,r]$ is actually principal $S = \mathcal{M}\ell$, while for such principal ideals it is easily seen that $L_S X = X\ell$ for all applications X of $\mathcal{M}$; thus for X to be S-skeletal merely means that all elements of X are fixed by the right action of ℓ. Suppose X is all fixed by ℓ; we must show that X is already an $[L,R]$-sheaf, so consider any morphism $[L,R] \xrightarrow{f} X$ of applications, which we must show comes from a unique complete element of X. The uniqueness is immediate, since if x,y are any two elements of X with the same $[L,R]$ part f, we have $xt = yt$ for all $t \in [L,R]$, but $t = \ell$ is such and we have already assumed X fixed by ℓ: thus $x = x\ell = y\ell = y$. For the existence of an x extending the partial element f, note that, while a general application X consists of a complicated interlocking system of "tacos", the skeletal condition means that these are all degenerated with $x = x\ell = xq = xr$, i.e. all three "edges" of any element x coincide; this implies also $xL = x\ell L = x\ell = x$ and similarly $xR = x$, leaving only the endpoint operators $x0$, $x \cdot 1$ acting possibly non-trivially: to sum up, such a skeletal $\mathcal{M}$-application is in essence just a directed graph. Now a partial element f defined only on the faces $[L,R] = \mathcal{M}L$ has in particular all its values fixed by ℓ due to the skeletal condition, so

$$f(L) = f(L)\ell = f(L\ell) = f(\ell)$$

$$f(R) = f(R)\ell = f(R\ell) = f(\ell)$$

the last being true because of the Aufhebung condition $R\ell = \ell$ in the definition of $\mathcal{M}$ itself. Thus the element $x = f(\ell)$ seems the likely candidate for a complete (degenerately) three-dimensional element whose restriction to the seven-element ideal $[L,R]$ could be f itself. Thus we try to show

$$\boxed{f(\ell)a = f(a)}$$

for all seven $a \in [L,R]$. For $a = L,R$ we have by the above

$$f(\ell)L = f(L)L = f(L)$$
$$f(\ell)R = f(R)R = f(R).$$

(both of course equal to $f(\ell)$). For the two constants $a = 0,1$ we have $f(\ell)a = f(\ell a) = f(a)$ since $\ell 0 = 0, \ell 1 = 1$. For the remaining three $a = \ell,q,r$ the case $a = \ell$ is tautologous, and for $a = r,q$ we have

$$f(\ell)r = f(\ell r) = f(\ell)$$
$$f(\ell)q = f(\ell)Lr = f(\ell Lr) = f(\ell)$$

so that we are reduced to showing that

$$f(r) = f(\ell) = f(q).$$

For this we need to use that f is defined also on the two-dimensional L,R since otherwise these could be three different edges (with the same endpoints $f(0), f(1)$) of the directed graph. But since $f(R) = f(\ell)$,

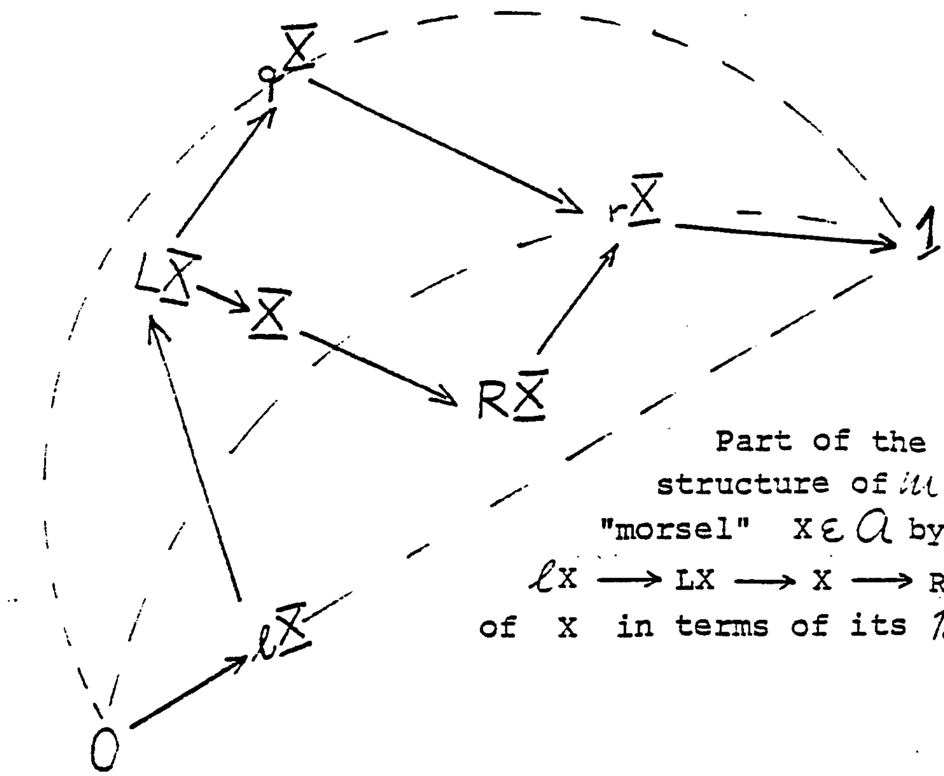
$$f(r) = f(Rr) = f(R)r = f(\ell)r = f(\ell r) = f(\ell)$$

and since $f(L) = f(\ell)$,

$$f(q) = f(Lr) = f(L)r = f(\ell)r = f(\ell r) = f(\ell)$$

so the proof is done.

Of course the above display does not show that $\mathcal{M}$ is a monoidal <u>category</u>, not just a graphic monoid; if $X \in \mathcal{A}$ is any "morsel", then the horizontal slice through the "taco" at X actually has canonical morphisms of $\mathcal{A}$ (indexed by $\mathcal{M}$), which are roughly the "Moore-Postnikov" analysis of X in case $\mathcal{A}$=combinatorial topology, as follows:



Part of the category structure of $\mathcal{M}$ revealed at the "morsel" $X \in \mathcal{A}$ by the analysis

$$\ell X \longrightarrow LX \longrightarrow X \longrightarrow RX \longrightarrow rX$$

of X in terms of its $\mathcal{B}$ and $\mathcal{G}$ reflections.

measuring how closely the various reflections of X (into the grasped stages $\mathcal{G}$, $\mathcal{B}$) succeed in approximating it.

<u>PROPOSITION 20</u> The "slice" obtained by omitting 0,1 from $\mathcal{M}$ is as a graphic monoid isomorphic to a six-element submonoid of the monoid of all order-preserving endomaps of a three-element linearly-ordered set; namely omitting 001, 002, 112, 122 from the latter corresponds to the former via $0 \mapsto \ell$, $1 \mapsto q$, $2 \mapsto r$. (Note that ℓ,q,r have <u>become</u> constants through this omission).

The proof is left to the interested reader.

NOTES

1) This research was not supported by any granting agency.

2) I am not a "Hegelian", since I reject Hegel's Objective Idealism. But Hegel's partly-achieved goal of developing Objective Logic (as a component of the laws of thought at least as important as the Subjective Logic commonly considered to be "all" of Logic) is in a way the program which the whole body of category theory has been carrying out within mathematics for the past 50 years. It was because of some discoveries in the foundations of homotopy theory that I began a few years ago the study of The Science of Logic, attempting to extract the "rational kernel" which, insofar as it truly reflects laws of thought, should be useful to us in investigations like the one summarized in this paper.

REFERENCES

[0] Hegel, G.W.F., The Science of Logic, 1812-1813-1816, another shorter version in his Encyclopedia, Heidelberg 1817, both now in several translations.

[1] Lawvere, F.W., Categories of spaces may not be generalized spaces, as exemplified by directed graphs, Revista Colombiana de Matemáticas, 20 (1986), 179-188.

[2] Lawvere, F.W., Qualitative distinctions between some toposes of generalized graphs, to appear in Contemporary Mathematics 92 (1989) (Proceedings of AMS Boulder Conference on Category Theory and Computer Science 1987).

[3] Schützenberger, M.P., Sur certains treillis gauches, C.R. Acad. Sc. Paris 224 (1947), 776-778.

[4] Kimura, N., The structure of idempotent semigroups I, Pacific J. Math. 8 (1958), 257-275.

[5] Kelly, G.M., and Lawvere, F.W., On the complete lattice of essential localizations, to appear in the Proceedings of the Louvain meeting in honor of René Lavendhomme's 60th birthday. (1989).

[6] Zaks, M., Doctoral Thesis on new aspects of simplicial sets, in preparation at Macquarie University, North Ryde, N.S.W. (Australia).