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State Categories and Response Functors

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Dedicated to Walter Noll

The concept of state, which was already included among Aristotelian categories, is logically presupposed by all science, yet becomes clear only after extended development of several particular sciences.

In recent decades, important advances in the understanding of the concept of state have been made by Walter Noll [1] and others [2], yet further simplification will allow still wider use of these advances. With the help of the modern theory of categories, due to Eilenberg and Mac Lane [3], many parts of geometry and analysis have been simplified through unification. The following considerations are offered in the conviction that category theory can also assist in unifying the diverse concepts of the foundation of continuum physics with each other and with other parts of mathematics.

Plato's observation (incorrectly attributed to Heracleitus) that "all is flux" and "nothing is", does not lead to exact science, for exact science requires sober consideration of precise, distinguishable states. On the other hand, states are just the way-stations in definite processes of change, so that neither states nor processes can do without the other. Further,

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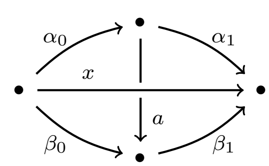
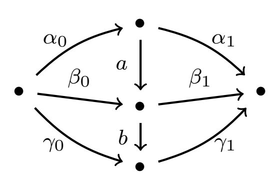
among processes there are special ones that are not processes at all. These observations are descriptive of the relationship of objects, morphisms, and identity morphisms in a mathematical category. Thus the theory of mathematical categories provides precise terminology and constructions for a significant part of the theory of states. For example, the notion of a **cyclic process**, beginning and ending at a given state, may be identified with that of **endomorphism** of a given object. But what particular structure and properties will categories, arising in this manner from continuum physics, enjoy? Can we hope to characterize such categories mathematically? It appears that they have very strong properties, some of which I shall try to explain in this paper.

Perhaps one day a (useful and not merely speculative) analysis of the concept of time will arise from a deeper study of state categories themselves, but (at least in the continuous case) we must here consider duration—as an additional structure on a category.

Definition 1. If x is a morphism in a category $\boldsymbol{\mathcal{X}}$, then the *interval* $\boldsymbol{\mathcal{I}}(x)$ is the category constructed as follows: An object of $\boldsymbol{\mathcal{I}}(x)$ is an ordered pair (α_0, α_1) of morphisms of $\boldsymbol{\mathcal{X}}$ for which $x = \alpha_1\alpha_0$ and a morphism $(\alpha_0, \alpha_1) \xrightarrow{a} (\beta_0, \beta_1)$ in $\boldsymbol{\mathcal{I}}(x)$ is any morphism of $\boldsymbol{\mathcal{X}}$ such that $\beta_0 = a\alpha_0$ and $\beta_1 a = \alpha_1$



(of course $\beta_1\, a\, \alpha_0 = x$ for any morphism $\alpha \xrightarrow{a} \beta$ in $\boldsymbol{\mathcal{I}}(x)$). If $\beta \xrightarrow{b} \gamma$ is another morphism in $\boldsymbol{\mathcal{I}}(x)$, then $\alpha \xrightarrow{ba} \gamma$ is easily verified to be another morphism in $\boldsymbol{\mathcal{I}}(x)$



thus defining the composition operation needed to make $\boldsymbol{\mathcal{I}}(x)$ a category.

Proposition 1. *If* $X_0 \xrightarrow{x} X_1$ *is a given morphism in a category* $\boldsymbol{\mathcal{X}}$*, then there is a canonical "forgetful" functor* $\boldsymbol{\mathcal{I}}(x) \longrightarrow \boldsymbol{\mathcal{X}}$ *which to any* $X_0 \xrightarrow{\alpha_0} A \xrightarrow{\alpha_1} X_1$ *with* $\alpha_1\alpha_0 = x$

assigns the object A in $\boldsymbol{\mathcal{X}}$. The category $\boldsymbol{\mathcal{I}}(x)$ always has an initial object $0_x = (1_{X_0}, x)$ and a terminal object $1_x = (x, 1_{X_1})$ which map to X_0, X_1 respectively via the canonical functor $\boldsymbol{\mathcal{I}}(x) \longrightarrow \boldsymbol{\mathcal{X}}$, while the unique morphism $0_x \longrightarrow 1_x$ in $\boldsymbol{\mathcal{I}}(x)$ maps to x.

We may sometimes write $A \in x$ to indicate that A is in the image of the canonical functor $\boldsymbol{\mathcal{I}}(x) \longrightarrow \boldsymbol{\mathcal{X}}$.

Definition 2. A functor $\boldsymbol{\mathcal{X}} \xrightarrow{d} \mathfrak{T}$ will be called a *duration* if for every morphism x in $\boldsymbol{\mathcal{X}}$ the induced functor $\boldsymbol{\mathcal{I}}(x) \xrightarrow{d} \boldsymbol{\mathcal{I}}(dx)$ is an isomorphism of categories. This definition will be used mainly when $\mathfrak{T}$ is a commutative monoid considered as a category with one object.

Remark 1. Our definition $\boldsymbol{\mathcal{I}}(x) \xrightarrow{\sim} \boldsymbol{\mathcal{I}}(dx)$ of duration functor d is equivalent to the requierement that d enjoy "unique lifting of factorization", i.e. $d(x) = t_1 + t_0 \Rightarrow$ there is a unique pair x_1, x_0 for which $x_1 x_0 = x$, $d(x_1) = t_1$, $d(x_0) = t_0$.

Proposition 2. *If $\boldsymbol{\mathcal{X}} \xrightarrow{d} \mathfrak{T}$ is a duration then each morphism $X \xrightarrow{x} X'$ in $\boldsymbol{\mathcal{X}}$ determines $t = d(x)$ in $\mathfrak{T}$ and a function (the succession of states determined by the process x)*

$$|\boldsymbol{\mathcal{I}}(t)| \xrightarrow{|x|} |\boldsymbol{\mathcal{X}}|$$

from the interval of length t in $\mathfrak{T}$ into the set of objects of $\boldsymbol{\mathcal{X}}$, whose value at the initial object of the interval is X and whose value at the terminal object of the interval is X'.

Proof 1. Compose the inverse of the functor induced by d with the canonical forgetful functor

$$\boldsymbol{\mathcal{I}}(t) \xleftarrow{\sim} \boldsymbol{\mathcal{I}}(x) \longrightarrow \boldsymbol{\mathcal{X}}.$$

Proposition 3. *If $\boldsymbol{\mathcal{X}} \xrightarrow{d} \mathfrak{T}$ is a duration and if $\mathfrak{T}$ is a category in which every morphism is an epimorphism, then $\boldsymbol{\mathcal{X}}$ also has the property that every morphism is an epimorphism.*

Proof 2. Suppose $X_0 \xrightarrow{\alpha} A \overset{x'}{\underset{x''}{\rightrightarrows}} X_1$ are such that $x'\alpha = x''\alpha$; we must show $x' = x''$. Since $d(x')d(\alpha) = d(x'')d(\alpha)$ and since $d(\alpha)$ is an epimorphism, we have $d(x') = d(x'')$. Denoting by $x = x'\alpha = x''\alpha$ the common value, we see that $\boldsymbol{\mathcal{I}}(x) \underset{\approx}{\xrightarrow{d}} \boldsymbol{\mathcal{I}}(dx)$ takes the two objects (α, x'), (α, x'') to the same object of $\boldsymbol{\mathcal{I}}(dx)$; hence these objects are the same, so that in particular $x' = x''$.

Proposition 4. *Let $\mathfrak{T}$ be a monoid (category with a single object) whose composition is commutative and written additively. Suppose $\mathfrak{T}$ satisfies the two cancellation properties*

$$t' + s = t'' + s \implies t' = t''$$
$$t + s = 0 \implies t = 0 \text{ and } s = 0.$$

*Then for any category $\boldsymbol{\mathcal{X}}$ which admits a duration functor $\boldsymbol{\mathcal{X}} \xrightarrow{d} \mathfrak{T}$ valued in $\mathfrak{T}$ (i.e. $d(\alpha_1, \alpha_0) = d(\alpha_1) + d(\alpha_0)$, $d(1_A) = 0$) one has that every morphism α in $\boldsymbol{\mathcal{X}}$ is **both** an epimorphism and a monomorphism but **not** an isomorphism **nor** an idempotent unless $\alpha = 1_A$ is an identity. That is, any α is both right cancellable and left cancellable, but*

$$\alpha_1\alpha_0 = 1_X \implies \alpha_0 = \alpha_1 = 1_X$$
$$\alpha\alpha = \alpha \implies \alpha = 1_A.$$

Proof 3. The first cancellation law for $\mathfrak{T}$ says that all morphisms in $\mathfrak{T}$ are epimorphisms, so by the previous proposition the same holds for $\boldsymbol{\mathcal{X}}$. Since the previous proposition can be dualized and since $\mathfrak{T}$ is commutative, it also follows that all morphisms in $\boldsymbol{\mathcal{X}}$ are monomorphisms. On the other hand if we have even a one-sided inverse (a retraction or section) then

$$\alpha_1\alpha_0 = 1_X$$
$$d(\alpha_1) + d(\alpha_0) = 0$$

so that, by the second cancellation property, $d\alpha_i = 0$ and hence $\alpha_i = 1_X$ since d is "locally" injective $\boldsymbol{\mathcal{I}}(1_X) \xrightarrow{\sim} \boldsymbol{\mathcal{I}}(d1_X) = \mathbf{1}$. Of course, the non-existence of non-identity cyclic processes α which are idempotent $\alpha\alpha = \alpha$ already follows from the fact that each α is an epimorphism.

Remark 2. The primary example of monoid $\mathfrak{T}$ for continuum mechanics is the set of non-negative real numbers under the operation of addition, so that the category $\boldsymbol{\mathcal{I}}(t)$ becomes isomorphic to the linearly ordered set $[0, t]$. For comparison we may also consider the discrete case where $\mathfrak{T}$ is the additive monoid of non-negative natural numbers; a striking difference between the two is that $t_1 > 0,\, t_2 > 0 \Longrightarrow \boldsymbol{\mathcal{I}}(t_1) \approx \boldsymbol{\mathcal{I}}(t_2)$ are isomorphic categories in the continuous case, whereas in the discrete case the size of an interval is intrinsic in the sense that a category can have at most one duration functor to the additive monoid of natural numbers. The properties stated in the above proposition are in the discrete case related to those which led Leroux [4] to a powerful generalization of the Möbius inversion formula.

Example 1. If $\mathfrak{T}$ is any category and if T_0 is any object, consider the category $\mathfrak{T}/T_0$ whose objects are morphisms $T \longrightarrow T_0$ in $\mathfrak{T}$ and whose morphisms are commutative triangles of the form

$$\begin{array}{ccc} T' & \longrightarrow & T \\ & \searrow \quad \swarrow & \\ & T_0 & \end{array}$$

in $\mathfrak{T}$. The obvious forgetful functor $\mathfrak{T}/T_0 \longrightarrow \mathfrak{T}$ is always a duration. If $\mathfrak{T}$ has left cancellation (more generally, if every morphism to T_0 in $\mathfrak{T}$ is a monomorphism) then $\mathfrak{T}/T_0$ has the stronger property of being a poset (that is, a category in which there is at most one morphism $X \longrightarrow X'$ for each given ordered pair of objects). In case $\mathfrak{T}$ is a monoid (so T_0 is the unique object) then $\mathfrak{T}/T_0$ is usually referred to as the divisibility of $\mathfrak{T}$; for example if $\mathfrak{T}$ is additive, $t' \longrightarrow t$ in $\mathfrak{T}/T_0$ iff $t' \geq t$, and the forgetful duration functor

$$\mathfrak{T}/T_0 \longrightarrow \mathfrak{T}$$

is usually denoted as "difference", which is a functor because $t'' \geq t' \geq t$ implies

$$(t'' - t') + (t' - t) = t' - t.$$

Incidentally, the category of categories seems the only reasonable context in which fundamental relationships between objects as disparate as posets and monoids become structure-preserving morphisms.

Example 2. Given a set $|\boldsymbol{\mathcal{X}}|$ and a suitable monoid $\mathfrak{T}$, a path-category can be constructed by considering the elements of the set as objects and functions $|\boldsymbol{\mathcal{I}}(t)| \xrightarrow{x} |\boldsymbol{\mathcal{X}}|$ as morphisms. Composition is defined by using a push-out property of

$$\begin{array}{ccccc} \mathbf{1} & \xrightarrow{1} & \boldsymbol{\mathcal{I}}(t_0) & & \\ {\scriptstyle 0}\downarrow & & {\scriptstyle j_0}\downarrow & \searrow^{\alpha_0} & \\ \boldsymbol{\mathcal{I}}(t_1) & \xrightarrow{j_1} & \boldsymbol{\mathcal{I}}(t_0+t_1) & \overset{x}{\dashrightarrow} & X \\ & & \xrightarrow[\alpha_1]{} & & \end{array} \qquad \text{where} \qquad \begin{aligned} j_0(s_0, s_1) &= (s_0, s_1 + t_1) \\ j_1(u_0, u_1) &= (t_0 + u_0, u_1). \end{aligned}$$

The push-out property (i.e. the unique existence of the indicated x) follows if $\boldsymbol{\mathcal{T}}$ has the property that all its intervals are **linearly ordered**; for if $w \in \boldsymbol{\mathcal{I}}(t_0 + t_1)$, then the alternative $w \leq t$ or $t \leq w$ allows the determination of $x(w)$ using either α_0 or α_1 as appropiate. Note that a subcategory (of a category with duration) which is moreover closed with respect to factorization, is again a category with duration; in practice, examples are such sub-categories, consisting of processes physically possible in certain contexts, of path-categories.

Smoothness properties such as differentiability are not preserved under the above push-out, yet in spite of this severe handicap the convenience of having a category (i.e. closure under composition) has traditionally been considered more compelling, leading precisely to the notion of "piecewise smooth" paths. The linear ordering appears again below.

Definition 3. A *freeze* of an object X in a category $\boldsymbol{\mathcal{X}}$ is an endomorphism f of X with the property that for any factorization $f = \alpha_1 \alpha_0$

$$\begin{array}{ccccc} & & X' & & \\ & {}^{\alpha_0}\nearrow & & \searrow^{\alpha_1} & \\ X & & \xrightarrow[f]{} & & X \end{array}$$

in $\boldsymbol{\mathcal{X}}$ one has $X' = X$ (i.e. that α_0, α_1 must also be endomorphisms).

Thus intuitively, a freeze is a process (for example a deformation) throughout which the state (or configuration) remains unchanged.

Proposition 5. *If $\boldsymbol{\mathcal{X}}$ is a category in which*

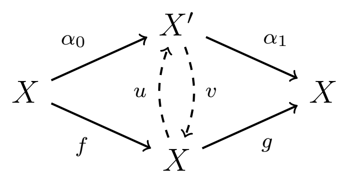
$$\boldsymbol{\mathcal{I}}(1_X) \xrightarrow{\sim} \mathbf{1} \text{ is a one element category for any object } X$$

and

$$\boldsymbol{\mathcal{I}}(x) \text{ is a linear order for any endomorphism } x,$$

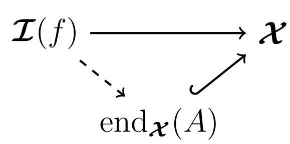
then the freezes of X form a submonoid of the endomorphism monoid $\boldsymbol{\mathcal{X}}(X, X)$ of all cyclic processes beginning and ending at X.

Proof 4. The first assumption is precisely the condition that 1_X has no non-trivial factorizations, so is certainly a freeze. We must show that the composition of two freezes f, g is a freeze. Consider any factorization $gf = \alpha_1\alpha_0$.



By the linear ordering, either there is a u such that $\alpha_0 = uf$ and $\alpha_1 u = g$, or there is a v such that $gv = \alpha_1$ and $f = \alpha_1\alpha_0$. In the latter case $X' = X$ because f is a freeze and in the former case $X' = X$ because g is a freeze. Hence gf is a freeze.

Note that in a monoid every morphism is a freeze, since in fact the condition that f be a freeze of A in any category $\boldsymbol{\mathcal{X}}$, is equivalent to the requirement that the canonical functor from the interval of f maps entirely into the **submonoid** of $\boldsymbol{\mathcal{X}}$ consisting of cyclic processes.



Proposition 6. *If $\boldsymbol{\mathcal{X}} \xrightarrow{d} \boldsymbol{\mathfrak{T}}$ is a duration with values in a monoid, then any section of d consists entirely of freezes of some object A of X.*

Proof 5. By a section of a functor d is meant a functor $\boldsymbol{\mathfrak{T}} \xrightarrow{\varphi} \boldsymbol{\mathcal{X}}$ such that $d\varphi = 1_{\boldsymbol{\mathfrak{T}}}$. If $\boldsymbol{\mathfrak{T}}$ is a monoid then $\varphi(0) = A$ is an object and φ parametrizes certain submonoid of

$\mathrm{end}_X(A)$ which is isomorphic to $\boldsymbol{\mathcal{T}}$. Now we use the special property that d is a duration to show that each $\varphi(t)$ is actually a freeze. The diagram

$$\begin{array}{ccccc} \boldsymbol{\mathcal{I}}(\varphi(t)) & \xrightarrow{\sim} \boldsymbol{\mathcal{I}}(d\varphi(t)) = & \boldsymbol{\mathcal{I}}(t) & \longrightarrow & \boldsymbol{\mathcal{I}}(\varphi(t)) \\ \downarrow & & \downarrow & & \downarrow \\ \boldsymbol{\mathcal{X}} & \xrightarrow[d]{\qquad} & \boldsymbol{\mathcal{T}} & \xrightarrow[\varphi]{\qquad} & \boldsymbol{\mathcal{X}} \end{array}$$

shows that $\boldsymbol{\mathcal{I}}(\varphi(t))$ maps into a monoid in $\boldsymbol{\mathcal{X}}$ since the functor goes via a monoid $\boldsymbol{\mathcal{T}}$.

A section φ of a monoid-valued duration need not meet all freezes at the object A which φ meets, although it must if all freezes at A commute. Indeed for any duration d and any two endomorphisms of A, if $d(f) = d(g)$ and $fg = gf$ then $f = g$.

For studies as diverse as control theory and the theory of material constitutive relations, it is necessary to consider explicitly the relation between processes of “deformation” which can be in principle freely carried out on “configuration” states and the processes on internal states which must necessarily accompany the former. This relation we will take to be that of “fibration”, to be defined presently. From a certain point of view the processes that it should be possible to “freely” carry out in any situation must include the freezes, although in fact the cost of the supplies necessary to keep a configuration frozen may be substantial.

Definition 4. A category $\boldsymbol{\mathcal{C}}$ equipped with a duration functor $\boldsymbol{\mathcal{C}} \xrightarrow{d} \boldsymbol{\mathcal{T}}$ to non-negative reals will be called a *configuration* category (and its morphisms referred to as *deformation* processes) in case for every object C of $\boldsymbol{\mathcal{C}}$ there is a given section φ_C of d for which $\varphi_C(0) = C$.

Remark 3. Of course in applications the configurations may include thermodynamical and other components as well as the purely mechanical aspects (such as infinitesimal metric) usually considered as configurations.

The duration may be considered as the supply of time required to carry out a process, in the sense of the general definition which follows:

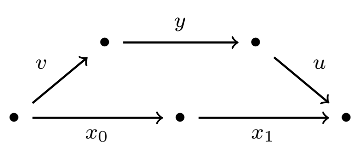
Definition 5. A functor $\boldsymbol{\mathcal{X}} \xrightarrow{Q} \boldsymbol{\mathcal{Y}}$ whose codomain is a commutative monoid will often be called a *supply*, and $Q(x)$ referred to as the supply of Q needed for the process x.

Proposition 7. *Let $p : \boldsymbol{\mathcal{V}} \longrightarrow \boldsymbol{\mathcal{W}}$ be a given functor between commutative monoids and let $\boldsymbol{\mathcal{X}} \xrightarrow{\Delta V} \boldsymbol{\mathcal{V}}$, $\boldsymbol{\mathcal{X}} \xrightarrow{W} \boldsymbol{\mathcal{W}}$ be supplies. Suppose $\boldsymbol{\mathcal{X}}$ has a duration to a commutative monoid $\mathfrak{T}$ whose intervals are linearly ordered and has $\boldsymbol{\mathcal{I}}(0) = \mathbb{1}$ and that $\boldsymbol{\mathcal{W}}$ is equipped with a submonoid $\boldsymbol{\mathcal{W}}_+$ of "non-negative" elements. Define*

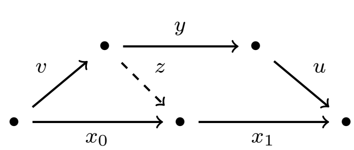
$$\boldsymbol{\mathcal{X}}_p = \{x \in \boldsymbol{\mathcal{X}} | \textit{for all } y, u, v, \textit{ if } x = uyv \textit{ then } W(y) \le p(\Delta V)(y)\}$$

where $\le$ is defined using $\boldsymbol{\mathcal{W}}_+$ in the usual manner. Then $\boldsymbol{\mathcal{X}}_p$ is a subcategory of $\boldsymbol{\mathcal{X}}$ with the same objects and the restriction of d to $\boldsymbol{\mathcal{X}}_p$ is still a duration.

Proof 6. If x is any identity process then any subprocess y (i.e. $x = uyv$) is actually $y = x$ since $\boldsymbol{\mathcal{I}}(x) \xrightarrow{\sim} \boldsymbol{\mathcal{I}}(0) = \mathbb{1}$; $(W)(\mathrm{id})$ and $(\Delta V)(\mathrm{id})$ are both zero, so $x \in \boldsymbol{\mathcal{X}}_p$. If $X \xrightarrow{x_0} X' \xrightarrow{x_1} X''$ are each in $\boldsymbol{\mathcal{X}}_p$ and if y is any subprocess of $x_1 x_0$,



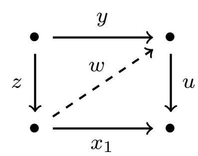
then by the linear ordering of $\boldsymbol{\mathcal{I}}(x)$ (where $x = x_1 x_0$) we can reduce the question whether $x \in \boldsymbol{\mathcal{X}}_p$ to three cases: namely $\boxed{\text{A}}$ $(v, uy) \le (x_0, x_1)$ or $\boxed{\text{B}}$ $(x_0, x_1) \le (v, uy)$; in the second case $\boxed{\text{B}}$, y is a subprocess of x_1, (hence y satisfies the inequality) while in the first case $\boxed{\text{A}}$ there is z for which



is commutative. Then z is a subprocess of x_0, hence in $\boldsymbol{\mathcal{X}}_p$. Thus there is a further alternative

$$\boxed{\text{A1}} : (z, x_1) \le (y, u) \text{ or } \boxed{\text{A2}} : (y, u) \le (z, x_1),$$

in the second case $\boxed{\text{A2}}$ of which y is a subprocess of z, hence in $\boldsymbol{\mathcal{X}}_p$, while in the first case $\boxed{\text{A1}}$ there exists w for which



is commutative. Then w is a subprocess of x_1, hence in $\boldsymbol{\mathcal{X}}_p$, so that y is a composite of w and z, both of which satisfy the inequality concerning the bound p on the average pressure.

But the calculation following shows that the processes satisfying the inequality form a subcategory:

$$\begin{aligned} W(y_1y_0) = Wy_1 + Wy_0 &\leq p(\Delta V)y_1 + p(\Delta V)y_0 \\ &= p((\Delta V)y_1 + (\Delta V)y_0) \\ &= p(\Delta V)(y_1y_0). \end{aligned}$$

Thus y satisfies the inequality as was to be shown.

Remark 4. If we assume $(\Delta V) = 0$ and $W(\varphi) \leq 0$ for freezes φ, (i.e. that work must be done **on** the system to keep it frozen), then $\boldsymbol{\mathcal{X}}_p$ will contain all freezes of $\boldsymbol{\mathcal{X}}$. We have described this (oversimplified) example partly to suggest the relativity of connectedness. Here we use the term “connected” in the naïve sense of category theory, so that the set of components of a category $\boldsymbol{\mathcal{X}}$ is obtained by applying the left adjoint of the full inclusion of the category of sets (as discrete categories) into the category of categories; thus X, X' are in the same component of $\boldsymbol{\mathcal{X}}$ if there exists a morphism $X \dashrightarrow X'$ in $\boldsymbol{\mathcal{X}}$ or in other words, X' is “accessible” from X. Alternatively, instead of taking the symmetric-transitive closure of the accessibility relation, one could instead require $X \substack{\dashleftarrow \\ \dashrightarrow} X'$ leading to smaller “strong” components. The following general remark applies not only to naïve components, but also to various refinements or limiting versions of the notion of accessibility. Although a state category $\boldsymbol{\mathcal{X}}_p$ may have several components and thus appear to be a conglomeration of state categories of several distinct materials, increasing the technology to $p' \geq p$ may lead to qualitative change in the sense that in $\boldsymbol{\mathcal{X}}_p$, it is possible to process one of these materials into another one. We would like to consider the inclusion $\boldsymbol{\mathcal{X}}_p \hookrightarrow \boldsymbol{\mathcal{X}}_{p'}$ as a morphism in the category of state categories and hence we do not make connectedness or accessibility part of our **general** axioms. In the situation described the inclusion would induce a surjective map from the set of components of $\boldsymbol{\mathcal{X}}_p$ to the set of components of $\boldsymbol{\mathcal{X}}_{p'}$.

Now we temporarily ignore the duration structure to define the central requirement of determinism. Consider a functor $\boldsymbol{\mathcal{X}} \xrightarrow{\pi} \boldsymbol{\mathcal{C}}$ in which we think of $\pi(X) = C$ as signifying that C is the configuration in which the internal state X manifests itself and similary $\pi(x) = \gamma$ signifies that γ is the deformation process which accompanies the internal-state-process x. We require that π is a (discrete op-)fibration in the sense of the following definition:

Definition 6. A functor $\boldsymbol{\mathcal{X}} \xrightarrow{\pi} \boldsymbol{\mathcal{C}}$ is a (discrete op-)*fibration* if whenever $C \xrightarrow{\gamma} C'$ in $\boldsymbol{\mathcal{C}}$, then for any X for which $\pi(X) = C$, there is a unique morphism x in $\boldsymbol{\mathcal{X}}$ for which both

$$\pi(x) = \gamma$$
$$\text{domain}(x) = X.$$

Denote by $\gamma \cdot X$ the codomain in $\boldsymbol{\mathcal{X}}$ of this unique x.

Proposition 8. *If π is a (discrete op-)fibration, then $X_1 \xrightarrow{x} X_2$, $\pi X_1 = \pi X_2 = C$ and $\pi(x) = 1_C$ implies $X_1 = X_2$ and $x = 1_X$ (i.e. the fibers of π are discrete as categories). One has*

$$1_C \cdot X = X$$

and whenever $C \xrightarrow{\gamma_0} C' \xrightarrow{\gamma_1} C''$ and $\pi(X) = C$, one has

$$\gamma_1 \cdot (\gamma_0 \cdot X) = (\gamma_1 \gamma_0) \cdot X$$

in the fiber over C''. In this way the category of all (discrete op-)fibrations with given base category $\boldsymbol{\mathcal{C}}$ is equivalent to the category

$$\mathcal{S}^{\boldsymbol{\mathcal{C}}}$$

of all set-valued covariant functors on $\boldsymbol{\mathcal{C}}$.

Proposition 9. *If $\boldsymbol{\mathcal{X}} \xrightarrow{\pi} \boldsymbol{\mathcal{C}}$ is a (discrete op-)fibration then π is in particular a "duration" ($\boldsymbol{\mathcal{I}}(x) \xrightarrow{\sim} \boldsymbol{\mathcal{I}}(\pi(x))$ for all x) and hence the composite $d\pi$ is a duration for $\boldsymbol{\mathcal{X}}$ whenever $\boldsymbol{\mathcal{C}} \xrightarrow{d} \mathfrak{T}$ is a duration for $\boldsymbol{\mathcal{C}}$.*

Remark 5. We may refer in a general way to functors $\boldsymbol{\mathcal{X}} \longrightarrow \boldsymbol{\mathcal{Y}}$ as "outputs", (which may include the measurements of "inputs") reserving the term "supply" to the case where $\boldsymbol{\mathcal{Y}}$ is an additive monoid and the term "response functor" for a situation in which both $\boldsymbol{\mathcal{X}}$, $\boldsymbol{\mathcal{Y}}$ are equipped with duration supplies to the same $\mathfrak{T}$ and the triangle

$$\begin{array}{ccc} \boldsymbol{\mathcal{X}} & \longrightarrow & \boldsymbol{\mathcal{Y}} \\ & \searrow \quad \swarrow & \\ & \mathfrak{T} & \end{array}$$

commutes.

A supply Q is said to be independent of path in case there is a codiscrete category $\boldsymbol{\mathcal{U}}$ (that is, one in which there is a unique morphism $U \longrightarrow U'$ for any ordered pair U, U' of objects) and some pair E, Δ of functors for which $Q = \Delta E$ in

$$\begin{array}{ccc} \boldsymbol{\mathcal{X}} & \xrightarrow{\quad Q \quad} & \boldsymbol{\mathcal{Q}}. \\ & {\scriptstyle E}\searrow \quad \nearrow{\scriptstyle \Delta} & \\ & \boldsymbol{\mathcal{U}} & \end{array}$$

It follows that the image of Q is a subgroup of the monoid $\boldsymbol{\mathcal{Q}}$.

A frequently arising situation is that in which a supply $Q : \boldsymbol{\mathcal{X}} \longrightarrow \boldsymbol{\mathcal{Q}}$ is absolutely continuous relative to another supply $S : \boldsymbol{\mathcal{C}} \longrightarrow \boldsymbol{\mathcal{A}}$ (which is considered also as a supply on $\boldsymbol{\mathcal{X}}$ by composing with a given fibration $\boldsymbol{\mathcal{X}} \xrightarrow{\pi} \boldsymbol{\mathcal{C}}$). This relationship involves the existence of a response

$$\boldsymbol{\mathcal{X}} \xrightarrow{\theta} \mathbf{Hom}(\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{Q}})$$

valued in a suitable category whose objects are additive maps (i.e. functors) from the additive monoid $\boldsymbol{\mathcal{A}}$ to the additive monoid $\boldsymbol{\mathcal{Q}}$, and is usually written

$$Q = \int \theta dS$$

meaning that for each morphism x in $\boldsymbol{\mathcal{X}}$, $Q(x)$ is "approximated" by

$$\sum \theta(X_i) \cdot S(x_i)$$

for finite partitions

$$\begin{array}{ccccccccccccc} & & X_1 & & & & X_2 & & & & X_3 & & & & \\ & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & & \\ X & & \xrightarrow[x_1]{} & & \bullet & & \xrightarrow[x_2]{} & & \bullet & & \xrightarrow[x_3]{} & & \bullet & \xrightarrow[x_4]{} & X' \end{array}$$

$(x = x_n x_{n-1} \cdots x_2 x_1, x_i \in X_i)$ which are sufficiently fine in the sense that

$$d(x_i) \leq S(X_i) \quad i = 1, \ldots, n$$

for a suitable positive function S on $|\boldsymbol{\mathcal{I}}(x)|$, d being the duration functor on $\boldsymbol{\mathcal{X}}$.

A constitutive relation over a configuration category $\boldsymbol{\mathcal{C}}$ involves a pair

$$\begin{array}{ccc} \boldsymbol{\mathcal{X}} & \xrightarrow{\theta} & \boldsymbol{\mathcal{Y}} \\ {\scriptstyle \pi}\downarrow & & \\ \boldsymbol{\mathcal{C}} & & \end{array}$$

where π is a (discrete op-)fibration and where θ is an output (response or supply). Some aspects $\eta\theta$ of θ may depend only on $\boldsymbol{\mathcal{C}}$ in the sense that there exists S with $S\pi = \eta\theta$. If all of the relevant response θ depends only on $\boldsymbol{\mathcal{C}}$, the state category $\boldsymbol{\mathcal{X}} \xrightarrow{\pi} \boldsymbol{\mathcal{C}}$ may as well be reduced to $\boldsymbol{\mathcal{C}}$ and we are dealing with one of the possible "elastic" responses $\boldsymbol{\mathcal{C}} \xrightarrow{\theta} \boldsymbol{\mathcal{Y}}$ whose configuration category is $\boldsymbol{\mathcal{C}}$. In general a survey of all possible constitutive relations over a given $\boldsymbol{\mathcal{C}}$ involves a range of possibilities for both the state category $\boldsymbol{\mathcal{X}} \xrightarrow{\pi} \boldsymbol{\mathcal{C}}$ fibered over $\boldsymbol{\mathcal{C}}$ and for the output. Let us consider separately some of the reasonable restrictions on each of these two aspects. The restrictions on constitutive relations which will be treated here are of a very logical nature and so provide at best a background for the more profound restrictions on the second aspect in the study of which Walter Noll was one of the pioneers. (I refer here to the derivation of equations of state from dissipation inequalities.)

For any given configuration category $\boldsymbol{\mathcal{C}}$ whose duration is valued in either the natural numbers or the non-negative real numbers, there is a natural restriction of a general nature that can be imposed on state categories over $\boldsymbol{\mathcal{C}}$.

Definition 7. Let $\mathrm{sh}(\boldsymbol{\mathcal{C}}) \hookrightarrow \mathcal{S}^{\boldsymbol{\mathcal{C}}}$ be the full subcategory consisting of all objects corresponding to (discrete op-)fibrations $\boldsymbol{\mathcal{X}} \xrightarrow{\pi} \boldsymbol{\mathcal{C}}$ which satisfy the following conditions: Let C be a given configuration, and suppose that for every deformation $C \xrightarrow{\alpha} A$ satisfying

$$d(\alpha) > 0$$

we are given a state X_α of configuration A in such a way that for any deformations $A \xrightarrow{\gamma} B$, $C \xrightarrow{\alpha} A$, $C \xrightarrow{\beta} B$ for which $\beta = \gamma\alpha$ and $d(\alpha) > 0$ (hence $d(\beta > 0)$ we have

$$\gamma \cdot X_\alpha = X_\beta;$$

then there is a unique state X of configuration C such that for all $C \xrightarrow{\alpha} A$ with $d(\alpha) > 0$

$$X_\alpha = \alpha \cdot X.$$

The above condition may be considered as a kind of "causal completeness": if all the possible "effects" $\alpha \cdot X$ (and hence $\theta(\alpha \cdot X)$ for all outputs θ) for all α with $d(\alpha) > 0$ are coherently present in the state category $\boldsymbol{\mathcal{X}}$, then the "cause" x itself should also be present and uniquely determined. It can be shown that the category $\mathrm{sh}(\boldsymbol{\mathcal{C}})$ is a topos [5], since the causal completeness condition is actually a "sheaf" condition with respect to a particularly simple example of Grothendieck covering which is induced via the duration from the positivity structure on the time translation monoid $\boldsymbol{\mathcal{T}}$. In the autonomous case, where $\boldsymbol{\mathcal{C}} \xrightarrow{d} \boldsymbol{\mathcal{T}}$ is an isomorphism, $\mathrm{sh}(\boldsymbol{\mathcal{C}}) \cong \mathrm{sh}(\boldsymbol{\mathcal{T}})$ consists, if $\boldsymbol{\mathcal{T}}$ is discrete, of invertible "discrete dynamical systems", but if $\boldsymbol{\mathcal{T}}$ is continuous, of possibly dissipative but semi-continuous actions of the additive monoid of non-negative reals. In the general non-autonomous case the duration functor $\boldsymbol{\mathcal{C}} \longrightarrow \boldsymbol{\mathcal{T}}$ induces an exact pair of adjoint functors (morphism of toposes) $\mathrm{sh}(\boldsymbol{\mathcal{C}}) \longrightarrow \mathrm{sh}(\boldsymbol{\mathcal{T}})$. Moreover, the categories $\mathrm{sh}(\boldsymbol{\mathcal{C}})$ of state categories are quite special among toposes even for arbitrary configuration categories $\boldsymbol{\mathcal{C}}$, as can be deduced from the work of Barel, Diaconescu, Freyd, Johnstone and Rosenthal (see Johnstone [5a]).

Theorem 1. *For any configuration category $\boldsymbol{\mathcal{C}}$ over a commutative monoid $\boldsymbol{\mathfrak{T}}$ with cancellation, there is a topological space $\mathcal{X}(\boldsymbol{\mathcal{C}})$ and a topological group G such that there are topos morphisms*

$$\mathcal{X}(\boldsymbol{\mathcal{C}}) \longrightarrow \mathrm{sh}(\boldsymbol{\mathcal{C}}) \longrightarrow \mathcal{B}_G$$

where the first is a surjective local homeomorphism and the second (to the Boolean classifying topos of G) is "localic" (meaning roughly that $\mathrm{sh}(\boldsymbol{\mathcal{C}})$ *can be reconstructed from an "internal topological space" with G-action).*

For discrete time $\boldsymbol{\mathfrak{T}} = \mathbb{N}$, the space $\mathcal{X}(\boldsymbol{\mathcal{C}})$ depends only on the graph

$$\boldsymbol{\mathcal{C}}_1 = \{\gamma \in \boldsymbol{\mathcal{C}} | d(\gamma) = 1\}$$

and is a generalization of Cantor space and Baire space (which themselves are in fact examples of $\mathcal{X}(\boldsymbol{\mathcal{C}})$ for particular $\boldsymbol{\mathcal{C}}$). For continuous time, there is a second "skeletal" inclusion $\mathrm{sh}(\boldsymbol{\mathcal{C}}) \xrightarrow{i_!} \boldsymbol{\mathcal{S}}^{\boldsymbol{\mathcal{C}}}$ left adjoint to the left adjoint i^* to the original inclusion i_*; hence $\mathrm{sh}(\boldsymbol{\mathcal{C}})$ is locally connected, with a "components" functor $\mathrm{sh}(\boldsymbol{\mathcal{C}}) \longrightarrow \boldsymbol{\mathcal{S}}$ assigning to each $\boldsymbol{\mathcal{X}}$ its set of components.

In fact, the conclusions in the above theorem require only the fact that all deformations are epimorphisms in $\boldsymbol{\mathcal{C}}$ (to obtain the local homeomorphic surjection from the space $\mathcal{X}(\boldsymbol{\mathcal{C}})$) and that every deformation is a monomorphism (to obtain the localic representation over $\mathcal{B}_G$); the existence of the further left adjoint in the continuous case is at bottom due just to the fact that every strictly positive duration can be expressed as the sum of two strictly positive durations. This particular conjunction of properties is only beginning to be studied in topos theory, yet in fact even our very general definition of duration functor and configuration category implies still more stringent requirements on the topos $\mathrm{sh}(\boldsymbol{\mathcal{C}})$, for example, those arising from the non-existence of non-trivial isomorphisms among the deformations and from the interaction between the causal completeness (sheaf) condition and the plenitude of freezes. Thus not only the already existing general topos theory, but also an emerging special branch thereof, provides a powerful background for the construction and comparison within $\mathrm{sh}(\boldsymbol{\mathcal{C}})$ of the various possible internal state categories which present the same configurational face.

A further condition on a state category $\boldsymbol{\mathcal{C}} \xrightarrow{\pi} \boldsymbol{\mathcal{C}}$, considered in effect by Walter Noll [1] as part of semi-elasticity, is that among all states above a given configuration C, there is a unique one X which is relaxed in the sense that

$$f \cdot X = X$$

for every freeze f at C. Of course, such a state is in particular one which admits freezes in the sense of the intrinsic structure of the category $\boldsymbol{\mathcal{X}}$ with duration. Relative to a given state category over $\boldsymbol{\mathcal{C}}$, one can define an elastic deformation in $\boldsymbol{\mathcal{C}}$ to be any deformation $C \xrightarrow{\alpha} C'$ such that if X is a relaxed state with $\pi X = C$, then $\alpha \cdot X$ is relaxed. For a fibration π satisfying Noll's condition, these form a subcategory $\boldsymbol{\mathcal{C}}_{\mathrm{elast}} \hookrightarrow \boldsymbol{\mathcal{C}}$ containing all configurations, and there is a functorial section λ of π over it

$$\begin{array}{ccc} & & \boldsymbol{\mathcal{X}} \\ & \overset{\lambda}{\nearrow} & \big\downarrow \pi \\ \boldsymbol{\mathcal{C}}_{\mathrm{elast}} & \hookrightarrow & \boldsymbol{\mathcal{C}} \end{array}$$

such that if φ is any section of d, φ actually has values in $\boldsymbol{\mathcal{C}}_{\mathrm{elast}}$, and $\lambda\varphi$ is a section of the duration $d\pi$ of $\boldsymbol{\mathcal{X}}$.

Now a standard example of fibration over $\boldsymbol{\mathcal{C}}$ is $\boldsymbol{\mathcal{C}}(C_0, -)$ for a particular configuration C_0; in considering this as a state category, those states which are of configuration C are the "finite histories" $C_0 \longrightarrow C$ which began at C_0 and which end at C; the fibration property is just the action by composition in $\boldsymbol{\mathcal{C}}$.

(Caution: In the case of discrete time this object is not in the subcategory $\mathrm{sh}(\boldsymbol{\mathcal{C}}) \hookrightarrow \mathcal{S}^{\boldsymbol{\mathcal{C}}}$ and its sheaf completion must be considered instead.) The object

$$\mathcal{H}_{\mathrm{fin}}(\boldsymbol{\mathcal{C}}) = \sum_{C_0 \in \boldsymbol{\mathcal{C}}} \boldsymbol{\mathcal{C}}(C_0, -)$$

has as its states all the finite histories. The famous Yoneda Lemma implies the following:

Proposition 10. *If $\boldsymbol{\mathcal{X}}$ is any object of* $\mathrm{sh}(\boldsymbol{\mathcal{C}})$*, there is a one-to-one correspondence between functors of state categories over* $\boldsymbol{\mathcal{C}}$

$$\boldsymbol{\mathcal{C}}(C_0, -) \longrightarrow \boldsymbol{\mathcal{X}}$$

and choices of a single state of $\boldsymbol{\mathcal{X}}$ of configuration C_0. Similarly, each functor of state categories over $\boldsymbol{\mathcal{C}}$

$$\mathcal{H}_{\mathrm{fin}}(\boldsymbol{\mathcal{C}}) \longrightarrow \boldsymbol{\mathcal{X}}$$

is uniquely determined by a function λ which assigns to each configuration C a state $\lambda(C)$ of underlying configuration C.

In case $\boldsymbol{\mathcal{X}}$ has the (fragment of) semi-elastic structure given by a function λ assigning to every configuration C the relaxed state $\lambda(C)$ of configuration C, we thus obtain a standard map

$$\begin{array}{ccc} \mathcal{H}_{\mathrm{fin}}(\boldsymbol{\mathcal{C}}) & \xrightarrow{\widetilde{\lambda}} & \boldsymbol{\mathcal{X}} \\ & \searrow \quad \swarrow_{\pi} & \\ & \boldsymbol{\mathcal{C}} & \end{array}$$

expressing one conception of how what **is** (German ‘Wesen’) results from what **was** (German ‘gewesen’). Using topology, limits, and further conditions on $\boldsymbol{\mathcal{X}}$, it may be possible to extend $\widetilde{\lambda}$ to some larger subcategory

$$\mathcal{H}_{\mathrm{fin}}(\boldsymbol{\mathcal{C}}) \hookrightarrow \ ? \ \hookrightarrow \mathcal{H}(\boldsymbol{\mathcal{C}})$$

of the category of infinite histories (which are defined [6] to be sections $\mathfrak{T}/0 \xrightarrow{h} \boldsymbol{\mathcal{C}}$ of d over the difference functor $\mathfrak{T}/0 \xrightarrow{\Delta} \mathfrak{T}$); the inclusion of $\mathcal{H}_{\mathrm{fin}}$ into $\mathcal{H}$ is defined using continuation into the past by constants. Special interest attaches to those semi-elastic state categories $\boldsymbol{\mathcal{X}}$ for which the map $\widetilde{\lambda}$ (or such an extension of it) maps surjectively to $\boldsymbol{\mathcal{X}}$, or at least densely in the sense that within a given family of preferred “continuous” outputs, a pair of continuous outputs agree on $\boldsymbol{\mathcal{X}}$ provided they agree when composed with $\widetilde{\lambda}$. Such a condition on $\boldsymbol{\mathcal{X}}$ seems to imply a drastic reduction of the range of possibilities for state categories compared to the proper class $\mathrm{sh}(\boldsymbol{\mathcal{C}})$. However, one must also consider the possibility of enlarging $\boldsymbol{\mathcal{C}}' \longrightarrow \boldsymbol{\mathcal{C}}$ (the notion of configuration (and deformation)), for example a purely mechanical $\boldsymbol{\mathcal{C}}$ may be enlarged to a thermomechanical $\boldsymbol{\mathcal{C}}'$. Thus in the extreme we could imagine being given only a fibration $\boldsymbol{\mathcal{X}} \longrightarrow \mathfrak{T}$ over the time monoid alone, an autonomous necessity without any given notion $\boldsymbol{\mathcal{C}}$ of deformation process. Deeper knowledge of $\boldsymbol{\mathcal{X}}$ may however be accompanied by the possibility of constructing a category $\boldsymbol{\mathcal{C}}$ of freely possible deformations, a fibration $\boldsymbol{\mathcal{X}} \xrightarrow{\pi} \boldsymbol{\mathcal{C}}$ and a functor $\mathcal{H}_?(\boldsymbol{\mathcal{C}}) \longrightarrow \boldsymbol{\mathcal{X}}$ which is surjective or dense.

Let us now consider the second aspect of a constitutive relation, namely the outputs, responses, and supplies (all taken as one here for simplicity). If the recipient category $\boldsymbol{\mathcal{Y}}$ for this has been specified, and also the configuration category $\boldsymbol{\mathcal{C}}$, then from a pragmatic point of view the "only" role of a state category

$$\begin{array}{ccc} ? & \dashrightarrow & \boldsymbol{\mathcal{Y}} \\ \downarrow & & \\ \boldsymbol{\mathcal{C}} & & \end{array}$$

is to account for observed correlations between processes in $\boldsymbol{\mathcal{C}}$ and in $\boldsymbol{\mathcal{Y}}$.[1] The above diagram suggests that given only a pair of categories $\boldsymbol{\mathcal{C}}$, $\boldsymbol{\mathcal{Y}}$ there is a universal state category $\mathcal{F}_{\boldsymbol{\mathcal{C}}}(\boldsymbol{\mathcal{Y}})$ to which all others can be reduced insofar as output into $\boldsymbol{\mathcal{Y}}$ is concerned. Indeed this follows from the construction, recently utilized by Grothendieck in homotopy theory [7], of an adjoint pair of functors

$$\mathbf{Cat} \underset{\mathcal{D}_{\boldsymbol{\mathcal{C}}}}{\overset{\mathcal{G}_{\boldsymbol{\mathcal{C}}}}{\leftrightarrows}} \mathcal{S}^{\boldsymbol{\mathcal{C}}}$$

in which the left adjoint $\mathcal{G}_{\boldsymbol{\mathcal{C}}}$ is the Grothendieck construction assigning to any set-valued functor on $\boldsymbol{\mathcal{C}}$ the total category of the associated (discrete op-)fibration; $\mathcal{F}_{\boldsymbol{\mathcal{C}}} = \mathcal{G}_{\boldsymbol{\mathcal{C}}} \circ \mathcal{D}_{\boldsymbol{\mathcal{C}}}$ is the endofunctor of **Cat** which we seek.

[1]Of course the pragmatic point of view contrasts with the materialist view which considers that a state category reflects the objective essence of the thing being studied, so will in particular remain the basis of responses valued in undiscovered Y', etc.

Theorem 2. *There is a fibration* $\mathcal{F}_{\boldsymbol{\mathcal{C}}}(\boldsymbol{\mathcal{Y}}) \longrightarrow \boldsymbol{\mathcal{C}}$ *over* $\boldsymbol{\mathcal{C}}$ *equipped with a canonical output* $\mathcal{F}_{\boldsymbol{\mathcal{C}}}(\boldsymbol{\mathcal{Y}}) \xrightarrow{\in} \boldsymbol{\mathcal{Y}}$ *to* $\boldsymbol{\mathcal{Y}}$ *such that for any fibration* $\boldsymbol{\mathcal{X}} \xrightarrow{\pi} \boldsymbol{\mathcal{C}}$ *and any functor* $\boldsymbol{\mathcal{X}} \xrightarrow{S} \boldsymbol{\mathcal{Y}}$*, there is a unique morphism of fibrations*

$$\begin{array}{ccc} \boldsymbol{\mathcal{X}} & \xrightarrow{\widehat{S}} & \mathcal{F}_{\boldsymbol{\mathcal{C}}}(Y) \\ & {\scriptstyle \pi}\searrow \quad \swarrow & \\ & \boldsymbol{\mathcal{C}} & \end{array}$$

for which $S = \in \circ \widehat{S}$*.*

In fact the set of states of configuration A in $\mathcal{F}_{\boldsymbol{\mathcal{C}}}(\boldsymbol{\mathcal{Y}})$ is the set

$$\mathbf{Cat}(A/\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{Y}})$$

of all functors η to $\boldsymbol{\mathcal{Y}}$ defined on the category of future deformations which start at A, and the action is defined by composing functors

$$A'/\boldsymbol{\mathcal{C}} \xrightarrow{\Sigma_\alpha} A/\boldsymbol{\mathcal{C}} \xrightarrow{\eta} \boldsymbol{\mathcal{Y}}$$

with the functor Σ_α of composing in $\boldsymbol{\mathcal{C}}$ with α. The canonical output $\in$ evaluates a state η at the object 1_A of $A/\boldsymbol{\mathcal{C}}$. If $\boldsymbol{\mathcal{Y}}$ and $\boldsymbol{\mathcal{C}}$ are equipped with duration functors and we consider responses S, a universal state category is again obtained by restricting the functors taken as states to be responses themselves. If instead $\boldsymbol{\mathcal{Y}}$ is a group, so that a universal state category with respect to supplies of $\boldsymbol{\mathcal{Y}}$ is sought, and if we exploit the fact that each category $A/\boldsymbol{\mathcal{C}}$ is actually a poset when all morphisms in $\boldsymbol{\mathcal{C}}$ are epimorphisms, then a universal state (A, η) can be considered as just a Y-valued mapping defined on the set of all deformations starting from A.

The above theorem thus clarifies the status of the "definition of state" given by Banfi and Fabrizio [8].

Combining the two extreme examples of state categories we have constructed, we can construct a third.

Proposition 11. *Given two categories* $\boldsymbol{\mathcal{C}}$*,* $\boldsymbol{\mathcal{Y}}$*, a map of fibrations over* $\boldsymbol{\mathcal{C}}$

$$\mathcal{H}_{\text{fin}}(\boldsymbol{\mathcal{C}}) \longrightarrow \mathcal{F}_{\boldsymbol{\mathcal{C}}}(\boldsymbol{\mathcal{Y}})$$

is determined by specifying for each C an arbitrary functor $C/\boldsymbol{\mathcal{C}} \xrightarrow{\eta_C} \boldsymbol{\mathcal{Y}}$.

The image of such a map in $\mathrm{sh}(\boldsymbol{\mathcal{C}})$ is a state category over $\boldsymbol{\mathcal{C}}$ with an output $\in$ to $\boldsymbol{\mathcal{Y}}$ satisfying both of the requirements that the states are determined by histories and distinguished by $\boldsymbol{\mathcal{Y}}$-outputs in the sense that if η, η' are two distinct states of the same configuration A, then there is some deformation $A \xrightarrow{\alpha} B$ such that $\in(\alpha \cdot \eta_1) \neq \in(\alpha \cdot \eta_2)$.

The choice of η_C for each C must thus be justified in order to render the above image construction of a state category for $\boldsymbol{\mathcal{C}}$, $\boldsymbol{\mathcal{Y}}$, somewhat canonical. Further investigation is needed to determine whether a choice of the type made by Noll, based on further axioms which relate a fading memory concept to relaxed states, can be characterized algebraically.

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