# STATE CATEGORIES, CLOSED CATEGORIES, AND THE EXISTENCE SEMI-CONTINUOUS ENTROPY FUNCTIONS

BY

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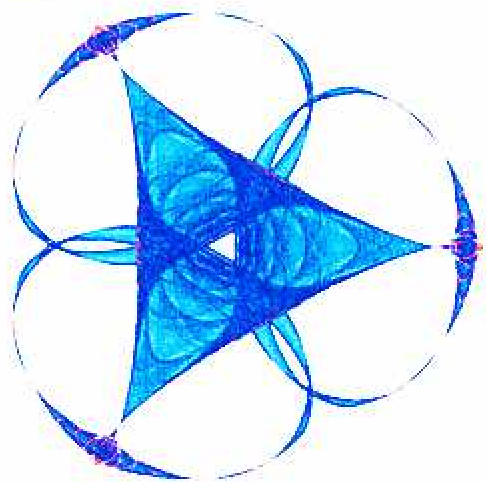

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State Categories, Closed Categories, and the Existence

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# Semi-Continuous Entropy Functions

F. William Lawvere

At the Minnesota Institute for Mathematics and its Applications in Summer 1983, M. Šilhavy and I independently found very similar simplifications of the hypothesis required to prove the 1974 general theorem of Coleman and Owen concerning the existence of upper semi-continuous entropy functions for general systems. My version, which involves at several places the simplifying concepts of category theory in order to unify the subject with other branches of mathematics in the hope of making the problems and results more widely comprehensible, is described briefly below.

Consider a category $\mathbb{X}$ whose objects will be called states and whose morphisms will be called processes. In applications there will often be a functor $\mathbb{X} \xrightarrow{\pi} \mathbb{C}$ to a "lower" category whose objects and morphisms are (generalized) "configurations" and "deformations" such that $\pi$ is a "discrete op-fibration", which expresses a kind of determinism appropriate to open systems and which implies that the morphisms of $\mathbb{X}$ are expressible as certain pairs $\langle \sigma, P \rangle$

where $\sigma$ is an object of $\mathbb{X}$ and $P$ is a morphism of $\mathbb{C}$. However, we will not use this extra structure, since discussion of that part of the interaction with the environment of $\mathbb{X}$ which is expressed by the entropy supply can be made independent of it. We also remark that in applications the category $\mathbb{X}$ (and $\mathbb{C}$ ) often has the unusual conjunction of properties that all morphisms are both monomorphisms and epimorphisms (i.e. left- and right-cancellable), but only the identity morphisms are isomorphisms (this would follow from the existence of a faithful embedding of $\mathbb{X}$ into a category of paths, with continuation as composition). But again we will make no use of this property of $\mathbb{X}$. Thus $\mathbb{X}$ is simply an abstract category, except that presently we will assume that the set of objects of $\mathbb{X}$ is endowed with a topology.

We will need two other categories, of a very simple concrete nature. Recall that any monoid can be considered as a category which has only one object and whose morphisms are the elements of the monoid, composition in the category being the given monoid operation; we will apply this construction to the additive monoid $\underline{\mathbb{R}}$ of extended real numbers. (This makes sense, as will be clarified below). Were we to consider heat supply rather than entropy supply, it would be appropriate to consider instead an additive monoid of signed measures on a hotness manifold.

The second kind of concrete category, intimately related to but completely distinct from the first kind, is that of a closed poset. Recall that any poset (set equipped with a reflexive and transitive relation $\leq$ ) may usefully be considered as a category whose objects are the elements of the set and whose morphisms are the pairs $\langle a_1, a_2 \rangle$ for which $a_1 \leq a_2$ ; for example the poset $\mathcal{V}$ of all extended reals with the usual order (or a set of signed measures on a hotness manifold, ordered using some judiciously specified cone of measures) will thus be construed as a category. For the "closed" structure of $\mathcal{V}$ see below.

A functor

$$\mathbb{X} \xrightarrow{\;\mathscr{s}\;} \mathbb{R}$$

will be referred to here as an <u>entropy supply</u>. Since $\mathbb{R}$ has only one object, we must have $\mathscr{s}(X_1) = \mathscr{s}(X_2)$ for any two states $X_1, X_2$ . The functoriality of $\mathscr{s}$ means that

$$\mathscr{s}(1_X) = 0$$

$$\mathscr{s}(\bar{\bar{P}} \cdot \bar{P}) = \mathscr{s}(\bar{\bar{P}}) + \mathscr{s}(\bar{P})$$

for any two composable processes $X_0 \xrightarrow{\bar{P}} X_1 \xrightarrow{\bar{\bar{P}}} X_2$ and for the identity process $1_X$ ("freeze of duration $0$ ") of any state $X$ . A function $S$ defined for the objects of $\mathbb{X}$ will be called a ("naive") entropy for $\mathscr{s}$ iff

$$S(X_1) + \mathscr{s}(P) \leq S(X_2) \quad \text{for all } X_1 \xrightarrow{P} X_2 \text{ in } \mathbb{X},$$

or briefly ( $\Delta S$ version)

$$\mathscr{s} \leq \Delta S$$

where $\Delta(S)(P) \underset{\mathrm{def}}{=} S(\mathrm{codom}\, P) - S(\mathrm{dom}\, P)$ makes sense with the judicious definition of subtraction below.

The entropy functions which we will construct will actually be representable (in the sense of Grothendieck). First we will consider the ("naive") case of M-representability, where for any two states $x_1, x_2$ we define

$$M(x_1, x_2) = \sup\{\Delta(P) \mid x_1 \xrightarrow{P} x_2 \text{ in } \mathbb{X}\}$$

Of course, any empty sup is $-\infty$. Then

$$0 \leq M(x,x)$$

for each state, and

$$M(x_0, x_1) + M(x_1 x_2) \leq M(x_0, x_2)$$

Proposition The following are equivalent

1) $0 = M(x,x)$
2) $M(x,x) < \infty$
3) (The "naive" Clausius property)

$\Delta(P) \leq 0$ for all endomorphisms $P$ of $x$ in $\mathbb{X}$.

For $b = M(x,x)$ is easily seen to satisfy $b + b = b$, and in our case of one-dimensional $\mathcal{V}$, this implies $b = 0$ or $b = \infty$ since $0 \leq b$. Endomorphisms in $\mathbb{X}$ are often called cyclic processes.

Proposition If we chose a state $x_0$ and define for any $x$

$$S(x) = M(x_0, x)$$

then $S$ is an entropy function for the supply $\mathcal{S}$ that was used to define $M$.

It is apparent that the fundamental transformation rule relating addition and subtraction which is used to obtain the $\Delta S$ version of the last proposition is the logical equivalence

$$\frac{a + b \leq c}{a \leq c - b}$$

which in terms of the category $\mathcal{V}$ means that the functor $(\ ) - b$ is right adjoint to the functor $(\ ) + b$. This situation appears in all branches of mathematics and is called a closed structure on $\mathcal{V}$, with (in our present example) + as the $\otimes$ and ( ) - ( ) as the Hom. Perhaps the best-known example has for $\mathcal{V}$ the category of vector spaces where the usual $\otimes$, Hom enjoy, for any three vector spaces a, b, c, the natural bijection of sets of linear transformations

$$\frac{a \otimes b \longrightarrow c}{a \longrightarrow \mathrm{Hom}(b, c)}$$

It is also required that $\otimes$ be associative and commutative (in a suitable functorial sense) and have a unit object (the space of scalars in the vector space example and 0 in the real $\mathcal{V}$ example).

we can define

$$a^* = \mathrm{Hom}(a, \mathrm{unit})$$

and in any closed category derive a canonical natural transformation

$$a^* \otimes b \longrightarrow \mathrm{Hom}(a, b)$$

which in the case of $\mathcal{V} = [-\infty, \infty]$ means

$$(-a) + b \leq b - a .$$

This natural transformation is not always an isomorphism; in the case of vector spaces it is an isomorphism for all $b$ iff $a$ is finite dimensional. Even in the case of the extended reals, the binary operation of subtraction is in general distinct from the indicated composite operation. In fact, we will have

$$\forall b \in \mathcal{V}[-a+b = b-a]$$

iff $-\infty < a < \infty$. To calculate the two sides of the inequality in case $a = \pm\infty$ note that

$$-\infty + b = -\infty \quad \text{for all } b$$

(including $b = \infty$ ) since the $\otimes$ functor $(\ ) + b$, being a left adjoint, must preserve all $\varinjlim$, i.e. in this case all sup's , including the empty sup. Thus $(-\infty) + \infty = -\infty$. On the other hand, as a special case of the $\otimes$, Hom adjointness we have for all $a, c$

$$\frac{c \leq \infty - a}{a + c \leq \infty}$$

But the condition below the line is true for all $a, c$, hence $c \leq \infty - a$ for all $c$, that is

$$\infty - a = \infty \quad \text{for all } a.$$

Thus the inequality

$$-a + b \leq b - a$$

is strict for $a = \infty$ and $b = \infty$ since then it just becomes

$$-\infty \leq \infty$$

Similarly one can show

$$-(-\infty) + b = \begin{cases} -\infty & \text{if } b = \infty \\ +\infty & \text{if } b > -\infty \end{cases}, \quad b - (-\infty) = \infty, \text{ all } b$$

To sum up, the requirement that $\mathcal{V} = [-\infty, \infty]$ be a closed category (for the usual notion $\leq$ of morphism and a tensor extending the usual $+$) forces the rational definition of the two binary operations of addition and subtraction which satisfy the adjointness transformation rules (more general than being inverse, which is not possible since groups cannot contain idempotent elements) which are needed for our application (and many others). Note that if morphisms were defined to be $\geq$ instead of $\leq$, a <u>different</u> meaning for $(\pm\infty) \pm (\pm\infty)$ would be forced; the resulting closed category of reals would be appropriate as a basis for a theory of metric spaces, rather than "entropic spaces" as here.

Closed categories are not only important as such, but even more as a basis for the theory of $\mathcal{V}$-categories, $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations. For example with $\mathcal{V}$ = vector spaces, one has linear categories, linear functors, linear natural transformations which are widespread in analysis, algebraic topology, homological algebra, etc. In general, if $\mathcal{V}$ is a closed category, a $\mathcal{V}$-category is any structure consisting of a set $\underline{X}$ of objects, a $\mathcal{V}$-object $M(x_1, x_2)$ for each pair of objects from $\underline{X}$, a "unit" $\mathcal{V}$-morphism

$$\mathrm{unit}_{\mathcal{V}} \longrightarrow M(x, x)$$

for each object $x$, and a "composition" $\mathcal{V}$-morphism

$$M(x_0, x_1) \otimes M(x_1, x_2) \longrightarrow M(x_0, x_2)$$

for each triple of objects; these are subject to unital and associative laws which need not concern us here since they are automatic when $\mathcal{V}$ is a poset. A $\mathcal{V}$-functor $(\underline{X}, M) \longrightarrow (\underline{X}', M')$ is any pair of mappings assigning an object $S(x)$ of $\underline{X}'$ to every object $x$ of $\underline{X}$ and a morphism in $\mathcal{V}$,

$$M(x_1, x_2) \xrightarrow{S_{x_1, x_2}} M'(S(x_1), S(x_2))$$

to each pair $(x_1, x_2)$ of objects of $\underline{X}$, satisfying certain axioms concerning preservation of unit and composition; in case $\mathcal{V}$ is a poset, the second mapping reduced to a condition on the first:

$$M(X_1, X_2) \leq M'(S(X_1), S(X_2)) \text{ in } \mathcal{V}$$

and the axioms are automatic.

Tautology If $s$ is an entropy supply functor, on an ordinary (Set-valued) category $\mathbb{X}$, then

$$M(X_1, X_2) = \sup\left\{ s(P) \,\middle|\, X_1 \xrightarrow{P} X_2 \right\}$$

defines a $[-\infty, \infty]$ -valued category structure on the set $\underline{X}$ of all objects of $\mathbb{X}$. Moreover an entropy function $S$ for $s$ is exactly a $\mathcal{V}$-functor

$$\langle \underline{X}, M \rangle \xrightarrow{S} \mathcal{V}.$$

where $\mathcal{V} = [-\infty, \infty]$ itself is made into a $\mathcal{V}$-valued category in the standard way with help of its Hom:

$$\mathcal{V}(a, b) = b - a$$

On any non-empty $\mathcal{V}$-category there are many $\mathcal{V}$-functors to $\mathcal{V}$, because there are the representable ones $M(X_0, -) : \langle \underline{X}, M \rangle \longrightarrow \mathcal{V}$ for any $X_0 \in \underline{X}$. In our case this representable entropy will have its value at $X_0$ $< \infty$ iff $s$ satisfies at $X_0$ the ("naive") Clausius property and in that case its value at any $X$ will be bounded above by $-M(X, X_0)$, which will in turn be finite provided $s$ itself is finite valued and $\exists X \cdots\!> X_0$.

Proposition $M(X_0, -)$ is the smallest among all the ("naive") entropy functions $S$ for which $\mathbb{O} \leq S(X_o)$.

Proof This is a special case of the Cayley-Dedekind-Grothendieck-Yoneda lemma, which states that for any $\mathcal{V}$-valued $\mathcal{V}$-functor $S$ on a $\mathcal{V}$-category $\overline{\underline{X}}$, $M$ and any object $X_o \varepsilon \overline{\underline{X}}$, there is a natural bijection

$$\frac{\mathrm{unit}_{\mathcal{V}} \longrightarrow S(X_o)}{M(X_o,-) \longrightarrow S}$$

between the set of $\mathcal{V}$-morphisms $\mathrm{unit}_{\mathcal{V}} \longrightarrow S(X_o)$ and the set of $\mathcal{V}$-natural transformations $M(X_o,-) \rightarrow S$ between the two indicated $\mathcal{V}$-functors $\langle X, M \rangle \rightrightarrows \mathcal{V}$.

Now we can move from the "naive" theory to the Coleman-Owen theory. Assuming that the set $\overline{\underline{X}}$ of states is endowed with a given topology, they introduced the idea of a (Coleman-Owen) entropy function $S$ which satisfies two further conditions

1) $S$ is upper-semicontinuous

2) The supply of entropy for any process whose domain is $X_1$ and whose codomain is sufficiently near $X_2$ is approximately bounded by $\Delta S$:

$$\forall \epsilon > 0 \, \exists W \ni X_2 \;\; \forall x \varepsilon W \;\; \forall X_1 \xrightarrow{P} X$$
$$\text{ᐃ}(P) \leq S(X_2) - S(X_1) + \epsilon,$$

(where $W$ denotes an open set) or in other words

$$\forall \epsilon > 0 \, \exists W \ni X_2 \;\; \forall x \varepsilon W$$
$$M(X_1, x) \leq S(X_2) - S(X_1) + \epsilon$$

or in still other words

$$\boxed{m(X_1, X_2) \leq S(X_2) - S(X_1)}$$

where

$$\boxed{m(X_1, X_2) \underset{\text{def}}{=} \inf_{W \ni X_2} \sup_{X \in W} M(X_1, X)}$$

They also introduced the strengthened (Coleman-Owen) Clausius condition on the entropy supply functor $\mathscr{s}$ that approximate endomorphisms should have approximately negative supply: for a given $X_0$

$$\forall \epsilon > 0 \, \exists U \ni X_0 \; \forall X \in U \; \forall X_0 \xrightarrow{P} X \qquad \mathscr{s}(P) \leq \epsilon$$

or in other words that

$$\forall \epsilon > 0 \; \exists U \in X_0 \; \forall X \in U \qquad M(X_0, X) \leq \epsilon$$

or in terms of the $m$ defined above simply

$$m(X_0, X_0) = 0$$

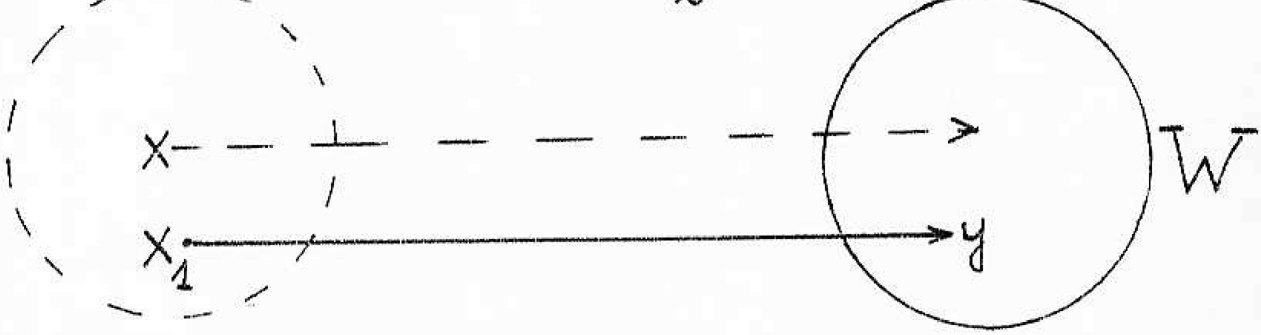
Now the above definition of $m$ is nothing but the standard upper-semicontinuous regularization (in the second variable) of $M$. Thus each function $m(X_0, -)$ is automatically upper-semicontinuous, and thus by our previous calculations will be a representable (Coleman-Owen) entropy function for $\mathscr{s}$ if only we can show that $\langle \underline{\bar{X}}, m \rangle$ is again a $\mathcal{V}$-category. Moreover, it will vanish at $X_0$ with the resulting boundedness properties at all $X$ from which $X_0$ is approximately accessible, if the (Coleman-Owen) Clausius property for $\mathscr{s}$ is true at $X_0$.

Thus (in effect following Coleman and Owen) we will concentrate on the question of whether $\langle \underline{\bar{X}}, m \rangle$ is a $\mathcal{V}$-category. The condition $0 \leq m(X,X)$ is clear, but for the "composition" law (reverse triangle inequality) we need an assumption. In Coleman and Owen 1974, conditions of accessibility, continuity of the action, and continuity of the supply were combined to obtain the desired result, but these continuities are with respect to variables which we for clarity have suppressed. It is possible to prove this central result without any assumptions of accessibility or finiteness by avoiding use of the "$+\epsilon$" method for proving statements of the form "$a \leq \sup_{i \in I} b_i$" and by taking as fundamental hypothesis the following <u>Semicontinuity Axiom</u> for $\mathbb{X} \xrightarrow{\mathscr{s}} \mathbb{R}$: If $\epsilon > 0$ and $m(x_1, x_2) > -\infty$ and if $W$ is a neighborhood of $x_2$, then there exists a neighborhood $V$ of $x_1$ such that for all $x \in V$ there exists a process $\bar{\bar{P}}$ starting at $x$ and ending in $W$ for which

$$m(x_1, x_2) \leq s(\bar{\bar{P}}) + \epsilon.$$

<u>Remark</u> The semicontinuity axiom would follow, provided relevant quantities were finite, from the hypothesis that for any $x_1 \xrightarrow{P} y \in W$ there is open $V \ni x_1$ such that for all $x \in V$ there is $x \xrightarrow{\bar{\bar{P}}} z \in W$ (for some $z$) such that

$$s(P) \leq s(\bar{\bar{P}}) + \frac{\epsilon}{2}$$




For we could (by implicit finiteness) chose $P$ with

$$\sup_{y \in W} M(x_1, y) \leq s(P) + \frac{\epsilon}{2}$$

In the Coleman-Owen 1974 work, more was (in effect) required of the $\bar{\bar{P}}$ above, roughly that for a $P: x_1 \longrightarrow y \in W$ for which $s(P) \approx m(x_1 x_2)$ a $\bar{\bar{P}}$ could be chosen with $s(\bar{\bar{P}}) \approx s(P)$ but moreover that $\tilde{\Pi}(\bar{\bar{P}})$ and $\tilde{\Pi}(P)$ should have the same time derivative, for $\mathbb{X} \xrightarrow{\tilde{\Pi}} \mathbb{C}$ a relevant fibration into a path category, as previously alluded to. Though such further properties of $\bar{\bar{P}}$ are both realizable and useful in many examples, they do not seem to be relevant to the narrow issue here being considered. In other respects our proof follows theirs (p. 12 and p.21) closely.

<u>Theorem</u> If $\mathbb{X} \longrightarrow \mathbb{R}$ satisfies the semicontinuity axiom, then $\underline{\overline{X}}, m$ is a $\mathcal{V}$-category, where $\mathcal{V} = [-\infty, \infty]$ and

$$m(x_1, x_2) = \inf_{W \ni x_2} \; \sup_{y \in W} \; \sup_{x_1 \xrightarrow{P} y} s(P)$$

<u>Proof</u> Since $0 \leq m(x, x)$ is clear, we need only show that for any $W \ni x_2$,

$$m(x_0, x_1) + m(x_1, x_2) \leq \sup_{z \in W} M(x_0 z)$$

By Yoneda's Lemma (for posets)

$$a \leq b \Longleftrightarrow \forall c \, [b \leq c \Longrightarrow a \leq c]$$

Thus a way to prove $a \leq \sup_{i \in I} b_i$ is to prove

$$\forall c \, [[\forall_i [b_i \leq c]] \Longrightarrow a \leq c$$

(since $\sup_{i \in I} b_i \leq c \Longleftrightarrow \forall i \in I \, [b_i \leq c]$).

Assume given any neighborhood $W$ of $x_2$ and any $c \in \mathcal{V}$ for which

$$M(x_0, z) \leq c \qquad \text{for all } z \in W$$

It must be shown that

$$m(x_0, x_1) + m(x_1, x_2) \leq c.$$

This is of course trivial if $c = \infty$ or $m(x_1, x_2) = -\infty$, so we may assume $c < \infty$ and that there is $a > -\infty$ for which

$$a \leq \sup_{y \in W} M(x_1, y)$$

Thus in particular we may assume that $\exists y \in W \, \exists x_1 \xrightarrow{P} y$
Then from the axiom there exists a neighborhood $V$ of $x_1$ having the there-specified property.
Now consider any $x \in V$ and any process $x_1 \xrightarrow{\bar{P}} x$. By the property of $V$ there exists (for any $\epsilon > 0$) $z \in W$ and $x \xrightarrow{\bar{\bar{P}}} z$ such that

$$m(x_1, x_2) \leq s(\bar{\bar{P}}) + \epsilon$$

Since $\bar{\bar{P}}$ and $\bar{P}$ can be composed, we have

$$s(\bar{P}) + s(\bar{\bar{P}}) = s(\bar{\bar{P}}\,\bar{P}) \leq c$$

since $z \in W$ and $\mathrm{dom}(\bar{\bar{P}}\,\bar{P}) = \mathrm{dom}(\bar{P}) = x_0$. Thus

$$s(\bar{P}) + m(x_1, x_2) \leq s(\bar{P}) + s(\bar{\bar{P}}) + \epsilon$$

Since the specific $X$ has now been eliminated from the second term, we have

$$s(P) + m(X_1, X_2) \leq c + \epsilon$$

for any $\bar{P}$ such that dom $(\bar{P}) = X_0$, codom $(\bar{P}) \varepsilon V$. Then

$$M(X_0, X) + m(X_1, X_2) \leq c + \epsilon$$

for all $X \varepsilon V$, and hence

$$m(X_0, X_1) + m(X_1, X_2) \leq c + \epsilon$$

but this holds for all $\epsilon > 0$, QED.

As for the choice topology on $\underline{\bar{X}}$ three remarks can be made. If the (naive) Clausius property holds for $s$, then

$$-d(x, y) \underset{\mathrm{def}}{=} M(x, y) + M(y, x) \leq M(x, x) = 0$$

defines a semi-metric on $\underline{\bar{X}}$; whereas if (stronger) $|M| < \infty$,

$$\delta(x, y) = \max(-M(x, y), -M(y, x))$$

defines a semimetric on $\underline{\bar{X}}$ with respect to which $S = M(X_0, -)$ itself (without passing to $m$) is even Lipschitz, so certainly upper semi-continuous [See Coleman-Owen ARMA 59, 1975]. For more on the relation between enriched functors and Lipschitz maps, see Lawvere 1973.

In case there is a given (discrete op-) fibration $\mathbb{X} \xrightarrow{\pi} \mathbb{C}$ to a category of paths in a configuration space, there is usually a nonnegative duration functor $\mathbb{C} \longrightarrow \mathbb{R}$ which by composition induces a duration or "(required) time supply" functor

$$\mathbb{X} \longrightarrow \mathbb{R} .$$

Now if dur is any duration functor and $A$ is any subset of the set of objects of the domain category, we could call $A$ "closed" iff for any morphism $P$ in the domain category, the set of all $t$ for which

$$\exists \bar{P}, \bar{\bar{P}} \left[ \bar{\bar{P}} \bar{P} = P, \quad \mathrm{dur}(\bar{P}) = t, \quad \mathrm{codom}(\bar{P}) \varepsilon A \right]$$

is closed in $[0, \mathrm{dur}(P)]$. In most examples where the domain category is an ample category $\mathbb{C}$ of continuous paths in a (generalized) configuration space, this definition recaptures the original topology on the latter space. But when dur is the (induced) time supply functor on a state category $\mathbb{X}$ , precisely the determinism of the constitutive relations expressed by the fibration property will usually prevent the class of those paths in $\underset{\sim}{\bar{X}}$ associated with morphisms in $\mathbb{X}$ from being "ample"; nonetheless the above definition of closed set gives a topology an $\underset{\sim}{\bar{X}}$ which it may be of interest to investigate. In particular, in what way must a given entropy supply functor $\mathcal{s}$ be related to a given time supply functor on the same category in order that the above semi-continuity axiom be verified relative to the defined topology?

Finally, since the key original motivation for the introduction of topology, approximate accessibility, stable Clausius property, etc. was the example of materials with memory in which $\mathbb{X}$ is (discrete op-) fibered over $\mathbb{C}$ by virtue of its objects being "histories" in $\mathbb{C}$ , it may be significant to observe that in many calculations regarding such examples, it is not so much the global open sets of $\overline{\underline{X}}$ which are involved as it is the topology restricted to a given fiber $\overline{\underline{X}}_C$ (= all histories which end at a given object $C \in \mathbb{C}$); possibly this observation could lead to an even further weakening of the semi-continuity axiom and to a narrowing of the weak approachability situation $m(X_1, X_2) > -\infty$ in which the axiom needs to be activated at all.

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