FUNCTORIAL REMARKS ON THE GENERAL CONCEPT OF CHAOS

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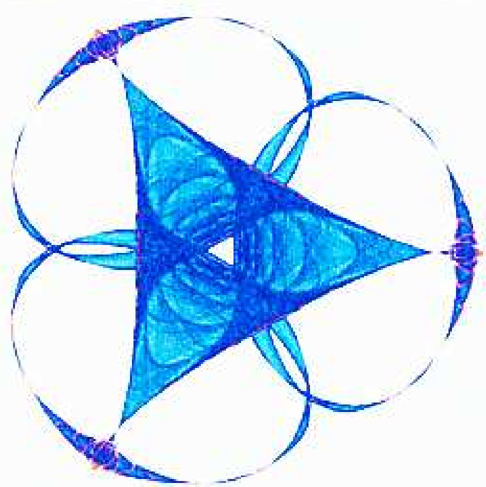

IMA Preprint Series # 87

July 1984

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

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Although the term "chaos" is employed in various ways in current dynamics literature, several instances [1], [2] of a precise usage have in common two features of a categorical nature: surjectivity of an induced map and right adjointness in the inducing process. Abstracting these features we propose a general definition and then note that there are examples which are rather different from the usual ones but which nonetheless have an intuitively "chaotic" quality.

Recall that a functor

$$\mathcal{X} \xrightarrow{U} \mathcal{Y}$$

between two categories is said to have a right adjoint

$$\mathcal{Y} \xrightarrow{H} \mathcal{X}$$

if there are natural transformations $1_{\mathcal{X}} \xrightarrow{\eta} HU$, $UH \xrightarrow{G} 1_{\mathcal{Y}}$ which induce bijections

$$\frac{UX \longrightarrow Y}{X \longrightarrow HY}$$

between the indicated sets of $\mathcal{Y}$-morphisms and $\mathcal{X}$-morphisms, for each X in $\mathfrak{X}$ and $Y \varepsilon \mathcal{Y}$. An important class of examples may be constructed as follows. Let $\mathcal{Y}$ be a suitable category of topological or differentiable spaces, and let T be a monoid in $\mathcal{Y}$. For example T could be the additive monoid of nonnegative reals or of nonnegative integers. Define $\mathfrak{X} = \mathcal{Y}^T$, the category whose objects are objects of $\mathcal{Y}$ equipped with actions of T, and whose morphisms are T-equivariant maps φ:

$$\begin{tikzcd} X \times T \arrow[r, "\varphi \times T"] \arrow[d, "\text{action}"'] & Y \times T \arrow[d, "\text{action}"] \\ X \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

commutative in $\mathcal{Y}$

We will write the actions of T multiplicatively and on the right. Define U to be the "underlying space" functor which forgets the action

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "U"] \arrow[d, equal] & \mathcal{Y} \\ \mathcal{Y}^T \arrow[ur, "\text{forget}"'] & \end{tikzcd}$$

As usual we will omit the symbol U from the notation when it causes no confusion. Then U has a right adjoint given by

$$H(Y) = Y^T$$

the internal function space (presumed to exist in $\mathcal{Y}$) for each space Y in $\mathcal{Y}$. The adjointness is demonstrated

in essence by the bijection
$$\frac{X \xrightarrow{\bar{\varphi}} Y^T}{X \xrightarrow[\varphi]{} Y}$$

between arbitrary maps φ and equivariant maps $\bar{\varphi}$ defined by

$$\varphi(x) = \bar{\varphi}(x)(u)$$ where u is the unit element of T

$$\bar{\varphi}(x)(t) = \varphi(xt)$$ where xt is the action given on $\underline{X}$

Here $\bar{\varphi}$ is equivariant if φ is given and we have equipped Y^T with the usual action by "translation":

$$(yt)(s) = y(ts) \qquad \text{for all } s \in T,\ y \in Y^T$$

Thus if $\underline{X}$ is a space equipped with a dynamical action of T and Y is a given space (without action) and if we consider a map $X \xrightarrow{\varphi} Y$ as an "observable" of state, then $\bar{\varphi}(x)$ is the function of time giving the progression of observed values of φ if we start in state x at $u =$ time 0. This $\bar{\varphi}$ is sometimes referred to as "symbolic dynamics", the points of Y being considered as symbols for the blocks into which φ divides the state space $\underline{X}$; then starting in state x, $\bar{\varphi}(x)$ is the T-sequence of blocks through which the dynamics takes the system.

<u>Definition</u> If $X \varepsilon \mathcal{Y}^T$, $Y \varepsilon \mathcal{Y}$, an observable $X \xrightarrow{\varphi} Y$ is T-<u>chaotic</u> iff the induced map $X \xrightarrow{\bar{\varphi}} Y^T$ is an epimorphism. More generally if $\mathfrak{X} \xrightarrow{U} \mathcal{Y}$ is a functor with right adjoint H and $UX \xrightarrow[\varphi]{} Y$, φ is U-<u>chaotic</u> iff $X \xrightarrow{\bar{\varphi}} H(Y)$ is an epimorphism.

That is (for those categories $\mathcal{Y}$ in which epimorphisms are surjective) $X \xrightarrow{\varphi} Y$ is a chaotic observable iff <u>every</u> T-sequence of symbols $T \xrightarrow{y} Y$ is realized as $y = \bar{\varphi}(x)$ for at least one state $x \varepsilon \underline{X}$. Refinements of this condition can be formulated by considering certain homomorphisms $T \to T'$ as periods and requiring the same condition after applying the corresponding change of action functors.

A standard example of a map φ often considered in this context is the adjunction

$$\mathbb{X} \longrightarrow \Pi_0(\mathbb{X}) = Y$$

where Π_0 is the "space of components" functor left adjoint to the inclusion $\mathcal{Y}_0 \hookrightarrow \mathcal{Y}$ of the category of prodiscrete spaces. In case $Y = \Pi_0(X)$ is actually finite and T is countable discrete, then $H(Y) = Y^T$ is a Cantor space.

Finally, in accord with loose usage we may say that a system $\bar{\mathbb{X}}$ in $\mathfrak{X} = \mathcal{Y}^T$ is "chaotic" in case there exists an $\mathfrak{X}$-subobject $\mathbb{X} \hookrightarrow \bar{\mathbb{X}}$ (that is, a T-invariant subspace) and a nontrivial space Y and a chaotic $\mathbb{X} \to Y$, or perhaps that $\Pi_0(\mathbb{X})$ is nontrivial and $\mathbb{X} \to \Pi_0 \mathbb{X}$ chaotic.

The extent to which the above general categorical definition expresses the informal notion of chaos can be illustrated by an example of a dual algebraic nature. Let $\mathcal{Y} = \mathcal{a}$ = the category of all commutative algebras over the reals; for example the function ring $C^\infty(Y)$ is an object of $\mathcal{a}$ for a manifold Y. Let $\mathcal{X} = \mathcal{a}'$ = the category of all such algebras A equipped moreover with derivations $A \xrightarrow{( )'} A$, that is $( )'$ is a real-linear map satisfying the Leibniz product rule, where the morphisms of $\mathcal{a}'$ are algebra homomorphisms which moreover commute with the given derivations. Of course any vector field supplies an example of an object in $\mathcal{a}'$. Then the forgetful functor

$$\mathcal{a}' \xrightarrow{U} \mathcal{a}$$

has the right adjoint H which assigns, to any algebra A, the algebra

$$H(A) = A[[t]]$$

of all formal (divided) power series with coefficients in A, equipped with the obvious formal derivation $\frac{d}{dt}$. The adjointness is verified as follows: If B is equipped with any given derivation and $B \xrightarrow[\varphi]{} A$ is an algebra homomorphism, then

$$B \xrightarrow[\varphi]{} A[[t]]$$

is defined by $\bar{\varphi}(f) = \sum \frac{\varphi(f^{(n)})}{n!} t^n$ where $f^{(n)}$ is defined by iterating the given derivation on B, and clearly $\bar{\varphi}(f') = \frac{d}{dt}\bar{\varphi}(f)$ as required for a'-morphisms.

Proposition: If $B = C^{\infty}(X)$, where X is a manifold, is equipped with the derivation induced by any given non-trivial vector field, then B has a map $B \xrightarrow{\varphi} \mathbb{R}$ which is chaotic relative to the adjunction $a' \leftrightarrows a$.

Proof: Let x be a point where the vector field is non-trivial and let $\varphi(f) = f(x)$ for all $f \varepsilon B$. Then $\bar{\varphi}(f)$ is the Taylor series of f. Borel's theorem states that <u>any</u> element of $H(\mathbb{R}) = \mathbb{R}[[t]]$ is the Taylor series at x of some smooth f. Thus $\bar{\varphi}$ is surjective, so that φ is chaotic relative to $a' \rightarrow a$, according to our definition.

Some other simple examples of right adjoints which may be of interest in this connection are determined by starting with a homomorphism $T' \longrightarrow T$ of monoids in a special category and considering the induced-action functor

$$S^T \xrightarrow{U} S^{T'}$$

whose right adjoint (generalizing ( ) when =1) is

$$H(Y) = \mathrm{Hom}_{T'}(T, Y)$$

the space of all T'-equivariant maps $T \longrightarrow Y$, where Y has some given T'-action and T' acts on T via translation and the given $T' \longrightarrow T$. For example if we consider the inclusion $\mathbb{N} \hookrightarrow \mathbb{R}^+$ of additive monoids, Y is essentially equipped with a single $\mathcal{S}$-endomorphism τ (the action of $1 \in \mathbb{N}$) and so is a "discrete time dynamical system", and the associated continuous-time dynamical system $H(Y)$ has as its states all $\mathcal{S}$-maps $\mathbb{R}^+ \xrightarrow{f} Y$ for which

$$f(t+1) = f(t)\cdot\tau \quad \text{for all} \quad t \geq 0$$

A morphism $\bar{X} \xrightarrow[\varphi]{} Y$, where $\bar{X} \in \mathcal{S}^{\mathbb{R}^+}$ is now required to satisfy the condition

$$\varphi(X\cdot t) = \varphi(X)\tau^n$$

in case $t = n$ is a whole number. [We remark that, as in many examples, our U has also a left adjoint $Y \longmapsto \mathbb{R}^+ \times_{\mathbb{N}} Y$, a tensor-like quotient space of $\mathbb{R}\times Y$, which gives a different notion of "continuous-time system associated to a discrete-time system".] Such a morphism φ is chaotic (in the present relative sense) if and only if every f of the kind described above are, for some $x \in \bar{X}$ of the form

$$f(t) = \varphi(xt) \quad \text{for all } t.$$

Of course there are standard examples of such φ, namely the evaluation at $0 : \bar{X} = H(Y) \longrightarrow Y$, but the novelty would be to find $\bar{X}$ constructed by other means.

In the case of a surjective $T' \longrightarrow T$, such as $\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ induced by a given period,

$$H(Y) \hookrightarrow Y$$

is simply the subspace of the T'-system Y consisting of points having that period, so for $X \xrightarrow[\varphi]{} Y$ to be "chaotic" relative to $T' \longrightarrow T$ simply means that all those points of Y <u>having the period in question</u> are values of φ.

As a final example, consider the problem of solving differential equations. From the point of view of "Synthetic Differential Geometry" [3] this can be considered as a special case of the special class of adjoints arising from a change of monoid $\langle D \rangle \longrightarrow \mathbb{R}$, but in any case, if $\mathcal{S}$ is a reasonable category of differentiable spaces, we can consider the category $\mathcal{Y}$ whose typical object Y is a space in $\mathcal{S}$ equipped with a vector field, and whose morphisms are smooth maps $Y_1 \longrightarrow Y_2$ which commute with the designated vector fields. Then if $\mathcal{X} = \mathcal{S}^{\mathbb{R}}$ is the category of reversible continuous-time flows, there is a functor $\mathcal{X} \xrightarrow{U} \mathcal{Y}$ given by differentiating each flow at $t = 0$. The right adjoint to U assigns to each such differential equation Y the space $H(Y)$ of all solution curves which are defined for all time. Thus for the adjunction map $H(Y) \longrightarrow Y$ to be "chaotic" in this context merely means that through each point of Y there is a solution curve which extends for all time. More generally if $\overline{X}$ is a flow and Y a vector field, a compatible map $X \xrightarrow{\varphi} Y$ is "chaotic" relative to the differentiation functor U iff every solution curve for Y is for some x the φ-image of the flow through x.

That the last few examples sound more like normal, reasonable behavior than like pathological chaos is explained in terms of the extent to which the functor U, or more particularly the monoid homomorphism $T' \longrightarrow T$, is an isomorphism: In the basic example $1 \to T$, $Y \mapsto H(Y) = Y^T$ is a big jump, so surjectivity of $\bar{\varphi}$ is harder to come by. In the case of $\langle D \rangle \hookrightarrow \mathbb{R}$, the functor U is not quite an equivalence but reasonable Y "think" it is in that $H(Y) \xrightarrow{\sim} Y$ due to the strong existence and uniqueness property; thus for such Y, the surjectivity of φ itself is sufficient for φ to be "chaotic".

[1] Saari, D.G., and Urenko, J.B, "Newton's method, Circle Maps, and Chaotic Motion", American Mathematical Monthly, Jan. 1984

[2] Coppel, W.A., "Maps on an Interval", IMA Preprint Series # 26, June 1983, Institute for Mathematics and its Applications, University of Minnesota.

[3] Kock, A. Synthetic Differential Geometry, LMS Lecture Note Series # 51, Cambridge University Press 1981.