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Categories in Continuum Physics

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INTRODUCTION

The articles collected in this volume reflect talks given at a workshop on Category Theory and the Foundations of Continuum Thermomechanics which was held at SUNY Buffalo in May 1982. The workshop permitted the beginning of more extensive exchange of ideas between groups of researchers which had previously had very little contact. W. Noll and W. Williams discuss here the foundations of the theory of material bodies, in particular the current status of Cauchy's stress theorem, while B. Coleman, M. Feinberg, R. Lavine, and D. Owen describe general contexts in which the existence of temperature and entropy have been established for rapidly deforming, unequally heated bodies. In apparently different directions, K.T. Chen and A. Frölicher discuss simplification of the foundation of infinite-dimensional differential geometry, as do A. Kock and G. Reyes with special attention to the axiomatics of that subject as a whole and to the variety of useful models of the resulting theory. The geometric theory is necessarily categorical, and I want to indicate below some of the advantages which may result if the abstract structures arising in thermomechanics are also explicitly recognized as categories. First I will indicate some of the reasons why a flexible geometric theory is demanded by continuum physics.

The mathematical background for theories of geometry, analysis, and continuum physics is usually considered to be the category of topological spaces or the category of Banach manifolds, with of course an infinite gradation of smoothness conditions needed (apparently) for various technical theorems. However, an essential construction in continuum physics is that of "function space", and the lack of well-behaved function spaces in those categories obscures the simplicity of geometrical or physically-motivated constructions and axioms. Yet, two centuries ago, many problems in the calculus of variations were correctly solved by mathematicians who, rather than defining a notion of "open subset" for their function spaces, took the notion of "path" as basic. Recognizing the great importance of contravariant concepts such as open set (or real function) does not commit us to take these as the _defining_ structure of a notion of space-in-general; they can be _derived_ concepts in a theory where the covariant concept of geometric _figures_ of some basic types, such as path, tangent vector, etc. are taken as primitive; theories of the latter kind can easily be constructed in which the unambiguous function space construction with good properties exists. It is with the construction of such categories that the articles of K. T. Chen, G. Reyes, and A. Frölicher are concerned. The "figures" used by Chen are called by him "plots", extending the less functorial charts of the usual atlases. Another significant aspect pointed out by these authors is that working in a category of smooth morphisms does not at all restrict one to considering only smooth subobjects in the usual manifold sense.

More precisely, Chen and Frölicher study two specific categories in which profound basic theorems about the smooth real line are used to construct notions of space which encompass many necessary examples that are not Banach manifolds, and are far simpler to describe from first principles than is the latter notion.

Kock and Reyes emphasize the axiomatic description of such categories as a whole. Axiomatizing a category as a whole promises to be part of the simplest approach to certain calculations. One exploits the discovery of Grothendieck that once the covariant "figure" attitude toward spatial objects is adopted, not only function spaces but also several other mysterious notions become easily manageable, such as infinitesimal paths, the spatial structure of the "set" of all linear subspaces of a given linear space, etc. Moreover, a "category of all spaces" can be construed as a "gros topos" which implies that fibered products and quotient constructions have exactness properties similar to those in the naive category of abstract sets but lacking in the usual categories of topological spaces or manifolds. The axiomatics at the category level is also valuable because there are many related categories which immediately come up. For example, if $\mathcal{X}$ is a gros topos of spaces and G is a group in $\mathcal{X}$ while S is an object of $\mathcal{X}$ then the categories $\mathcal{X}/S$ of S-parameterized families of spaces, $\mathcal{X}^G$ of actions of G on spaces in $\mathcal{X}$, and $\mathcal{X}^G/S$ (of central interest in bifurcation theory) are all categories which satisfy the same axioms as $\mathcal{X}$, as does (a reasonable determination of) the category of all objects of $\mathcal{X}$ equipped with affine connection.

Let me be more explicit about the role of the cartesian-closed property of a category (a <u>topos</u> is a cartesian-closed category in which moreover the notion of subobject is representable by a "truth-value" object). Let E denote ordinary physical space, T a space which represents the notion of time, and B a space which represents a particular body. Then a particular motion of B may be represented as a map

$$B \times T \longrightarrow E$$

which is the correct way if we want to compute by composition how particles of the body at various times experience the values of some field defined on space. However, it is also necessary to construe the same motion as a map

$$T \longrightarrow E^B$$

where the space E^B of (possibly singular) placements of the body is itself independent of T or a particular motion, if we want to compute by composition the temporal variation of quantities like the center of mass $E^B \longrightarrow E$ of B . Still a third version

$$B \longrightarrow E^T$$

of the same motion, where the space E^T of paths in space exists independently of B , is a necessary step if we want to compute by composition the velocity field on B induced by the motion. The possibility of passing freely among these three versions of the "same" map is obviously more fundamental for phrasing general axioms and concepts of continuum physics than is the precise determination of the concept of spaces-in-general (of which E,T,B are to be examples), yet these transformations are <u>not</u> possible for the commonest such determinations (for example Banach manifold). The general possibility of such transformations within a given category is called cartesian closure; a category with finite cartesian products (including an empty product 1) is cartesian closed if for any two objects A,Y there is another Y^A such that for any object X there is a natural bijection

$$\frac{X \longrightarrow Y^A}{A \times X \longrightarrow Y}$$

of maps (= morphisms in the given category). In particular any functional $Y^A \longrightarrow Z$ when composed with any $I \longrightarrow Y^A$ gives a map $I \longrightarrow Z$. A lemma proved by Grothendieck and by Yoneda (and in special cases by Cayley and Dedekind) says in effect that an object in a category is entirely determined by all morphisms into it from all possible objects. But in many categories there is a small set of objects such that morphisms from them alone into an arbitrary object determine the latter. Such special objects I might be called generic figures, and morphisms $I \xrightarrow{x} X$ particular figures of type I in X . If the I's are adequate in the sense [Isbell] just alluded to, a morphism $X \xrightarrow{f} Y$ is determined by the abstract mapping $x \longmapsto fx$ of figures in X into figures in Y ; and more importantly, such an abstract mapping is "smooth" (i.e. comes from an actual morphism $X \longrightarrow Y$ in the category) if only it is natural, i.e. satisfies the property $f(xa) = (fx)a$ for all $I' \xrightarrow{a} I$ between figures only (one might say that the mapping preserves generalized incidence relations). Such reasoning is sufficient to rigorously support the calculations of the Calculus of Variations, taking intervals as the adequate generic figures: to discuss the smoothness of a purported functional

$$Y^A \xrightarrow{J} Z$$

it suffices to check its compositions with all $I \xrightarrow{\bar{v}} Y^A$, but the latter are equivalent to $A \times I \xrightarrow{v} Y$ which are of lower type.**) Since the v are the origin of the term "variation" in "Calculus of Variations" we may say that the combination of the notion of cartesian closed category with that of generic figure to yield a determination of "space-in-general" is a natural development of those 18th century ideas. For convincing substantiation of these apparently simple-minded remarks, see the work of Chen and Frölicher.

In the articles of Kock and Reyes an important additional feature is the consideration of further generic figure types such as D , the tangent vector. This object is explicitly definable in terms of the smooth line R , as the subspace of R consisting of all $t \in R$ for which $t^2 = 0$. This object D can be non-trivial without changing the morphisms $R^n \longrightarrow R^m$, which remain the usual smooth maps (or the usual analytic or algebraic maps in other models of the axioms). In fact, D can be big enough that

$$R^D \xleftarrow{\sim} R \times R$$

in the sense that every smooth function defined near 0 restricts to equal a <u>unique</u> affine-linear function on D . Identifying the function space X^D with the tangent-bundle of an arbitrary space X in the category, a vector field $X \longrightarrow X^D$ becomes equivalent, via the fundamental transformation, to an action

$$D \times X \xrightarrow[\xi]{} X \qquad \xi(0,x) = x$$

of D on X . An obvious way for such to arise is to restrict some flow (action of the additive group of R)

$$R \times X \longrightarrow X$$

**) The smoothness of morphisms whose domain is a product $A \times I$ can be analyzed by testing against arbitrary $I \longrightarrow A \times I$ as is further explained in Frölicher's article.

from R to D , such restriction defining an instance of differentiation. This gives rise to a pair of adjoint functors

$$\text{Vector-fields} = \mathcal{X}^{D} \underset{\longrightarrow}{\overset{(\ )^{\bullet}}{\longleftarrow}} \mathcal{X}^{R} = \text{Flows}$$

where the right adjoint to differentiation takes X, ξ to

$$\mathrm{Hom}_{D}(R, X)$$

the space of solution curves. The adjunction map of flows, evaluation at 0 , is an isomorphism when the vector field given on X satisfies the existence and uniqueness properties as an ODE. It is worth pointing out that the definition of morphism ψ in the category $\mathcal{X}^{D}$ of vector fields, namely commutativity of

$$\begin{array}{ccc} D \times X & \xrightarrow{D \times \psi} & D \times Y \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \eta} \\ X & \xrightarrow[\psi]{} & Y \end{array}$$

is equivalent by the fundamental transformation to commutativity of

$$\begin{array}{ccc} X^{D} & \xrightarrow{\psi^{D}} & Y^{D} \\ {\scriptstyle \bar{\xi}}\downarrow & & \downarrow{\scriptstyle \bar{\eta}} \\ X & \xrightarrow[\psi]{} & Y \end{array}$$

which makes sense even in the usual category of manifolds where the tangent bundles are not function spaces in the way they are here. To deal with non-autonomous flows, the monoid R can be replaced by a suitable small category whose objects are instants of time.

Some fundamental concepts of continuum physics can be formulated in such categories before determination of which category is most appropriate for special calculations, or even in some cases before distinguishing between vector fields $\mathcal{X}^{D}$ and flows $\mathcal{X}^{R}$. For example, Muncaster's clarification [Muncaster] of the general problem of deriving coarse theories from fine theories is (backed up by his study of several key cases) in essence the following (with M = D or M = R): Let X be an object of fine states equipped with a dynamics, i.e. $X \in \mathcal{X}^{M}$ with action ξ , and let $Y \in \mathcal{X}$ be an object of coarse states with an $\mathcal{X}$-map $X \xrightarrow{\pi} Y$ (often an averaging process of some sort). It is desired to find dynamical actions η on Y which are somehow compatible with ξ and π , but experience shows that this should <u>not</u> mean that π becomes an $\mathcal{X}^{M}$ morphism for the choice of η ; π may preserve balance laws but not constitutive relations, both ingredients being involved in specifying η . Call an M-action η a non-linear "eigenvalue" of ξ (with multiplicity Y), if we can find an injective M-morphism G (in the direction opposite to π)

$$\begin{array}{ccc} M \times Y & \xrightarrow{1_M \times G} & M \times X \\ \eta \downarrow & & \downarrow \xi \\ Y & \xrightarrow[G]{} & X \end{array} \qquad G\eta = \xi(1_M \times G)$$

where such G could be called a non-linear "eigenvector". A gross determiner is such a non-linear eigenvector which (interprets coarse states as special fine states as above and) satisfies the constraint $\pi \circ G = 1_Y$ (which of course forces G to be injective). It is to be expected that there may be many such eigenpairs η, G for a given fine theory X, ξ and given "averaging" π to coarse states Y ; but the gross determiner G uniquely determines the coarse theory η (as with the usual linear eigenvalues).

The adjointness above (between differentiation of flows and solution curves) actually holds for any change of operator domain (like $D \subset R$), for example for the change $1 \xrightarrow{0} \mathbb{N}$, where 1 is the trivial domain and $\mathbb{N}$ is the additive monoid of natural numbers. An $\mathbb{N}$-action is equivalent just to a single endomorphism thought of as the change of state in one time unit. The adjunction in this case

$$\mathcal{X} \underset{(\ \ )^{\mathbb{N}}}{\overset{\text{forget action}}{\rightleftarrows}} \mathcal{X}^{\mathbb{N}}$$

is simply this: if Y is any object, then the object $Y^{\mathbb{N}}$ of sequences in Y has a standard action usually called "shift". If X is an object equipped with any endomorphism ξ , then an equivariant map

$$X \xrightarrow{\overline{\psi}} Y^{\mathbb{N}}$$

is entirely determined by an ordinary map

$$X \xrightarrow{\psi} Y$$

(namely $\psi(x) = \overline{\psi}(x)_0$) through the formula

$$\overline{\psi}(x)_n = \psi(\xi^n x) \qquad \text{all } n \ ,$$

and for arbitrary ψ , the $\overline{\psi}$ so defined is in fact equivariant. This is sometimes referred to as "symbolic dynamics", Y being considered as blocks into which X is divided by ψ , and the $\overline{\psi}$ thus assigning to a state x the sequence of blocks through which the dynamics ξ

takes x . In case Y is finite, $Y^{\mathbb{N}}$ is a Cantor Space in $\mathcal{X}$. The basic concept (which can be further ramified) of the currently popular "chaos" is that of a morphism ψ for which the induced $\overline{\psi}$ to the right adjoint is <u>surjective</u> (i.e. ψ observes so little of the states x that <u>any</u> given sequence of blocks can occur for some choice of x).

The important distinction between intensive and extensive quantities can also be exemplified in any category $\mathcal{X}$ of the kind under consideration. While these terms, of philosophical origin, are costumarily employed only in thermodynamics, (contrasting temperature, pressure, and density with energy, volume, and mass), they are actually applicable throughout continuum physics and indeed in mathematics generally. While their importance is most evident when interpreted relative to a given body B , it is useful to consider intensive and extensive quantities relative to any space X . The existence of this philosophical terminology is moreover fortunate because terms like "Radon measure" , "Schwartz distribution" , "singular homology class", etc. prejudice the issue in that they are but realizations of the general notion of extensive quantity resulting from various particular determinations of the spatial category $\mathcal{X}$. If we suppose given a ring object R in $\mathcal{X}$ then a basic notion of intensive quantity relative to an object X is that of a morphism $X \longrightarrow R$. Thus R^X is the space of intensive quantities on X . Two distinguishing features of intensive quantity are <u>contravariance</u>: there are

$$R^X \longleftarrow R^Y$$

induced for any $X \longrightarrow Y$, and <u>multiplicativity</u>: R^X is again a <u>ring</u> object (not just an R-linear space) and the foregoing induced morphisms $R^X \longleftarrow R^Y$ are ring homomorphisms (not merely R-linear). By contrast extensive quantity $M(X)$ should be <u>covariant</u>:

$$M(X) \xrightarrow[\psi!]{} M(Y)$$

exists induced by any $X \xrightarrow{\psi} Y$ and is <u>linear</u> but <u>not</u> multiplicative. (A ring structure on $M(X)$ can often be defined via convolution if X is a group or monoid, but not if X is just a space). However, the linearity of M is stronger than just R-linearity in the sense that $M(X)$ is actually a module over the ring R^X and the induced maps $M(X) \longrightarrow M(Y)$ are linear with respect to that, i.e. for $X \xrightarrow{\psi} Y$

$$\psi!(g\,\psi\cdot m) = g\cdot\psi!(m)$$

for any $g \in R^Y$, $m \in M(X)$. Moreover, there is a pairing

$$R^X \times M(X) \xrightarrow{\int_X} R$$

between intensive and extensive quantity which satisfies naturality equations when X is varied, and

$$\int_X f d(g \cdot m) = \int_X (fg) dm \quad \text{for all } f \ .$$

All these listed properties of M follow easily if we simply define

$$M(X) = \mathrm{Lin}_R(R^X, R)$$

the space of $\mathcal{X}$-smooth linear functionals, where in general $\mathrm{Lin}_R(V,W)$ is the subobject of W^V defined by the linearity equations, for any R-linear spaces V,W in $\mathcal{X}$. Then $\int$ is simply evaluation and the module structure is defined by the above fg equation. This definition of M can be proved (in all the categories considered by Chen, Frölicher, Kock and Reyes) to coincide with usual distributions with compact support, even though smoothness in those categories is defined covariantly in terms of generic figures rather than contravariantly in terms of open sets or seminorms. In other toposes it would agree rather with topological or abstract measure theory. The module structure expresses the important concept of density: if m,v are two quantities extensive with respect to the same space, then

$$\frac{dm}{dv} = \rho$$

simply means that ρ is an intensive quantity such that $m = \rho \cdot v$ for the module structure. Leaving to particular determination of the category the question of precisely which pairs m,v admit such a ρ, we may observe that its "uniqueness" (in a natural sense) is tautological: if $\rho_1 \cdot v = \rho_2 \cdot v$, then $(\rho_1 - \rho_2) \cdot v = 0$ i.e. $\rho_1 \equiv \rho_2$ modulo v. It will be seen that the covariant functoriality of extensive quantity is an essential background feature permitting Feinberg and Lavine's passage from states to thermodynamic states and Schanuel's passage from the size of a potato to its polynomial measure on space.

The categorical striving for unity and simplicity may also lend clarification to problems of continuum physics in another way. Beyond the background questions where the nature of categories like spaces-in-general, flows-in-general, etc. is studied, there is also the observation that, like many other branches of mathematics, thermomechanics deals with structures which themselves may sometimes usefully be seen as ("small") categories. The article by D. Owen is a case in point.

The general theory of thermodynamical systems developed by him and Coleman has features equally applicable both globally to a body as a whole and infinitesimally to body elements, and so there arises the crucial problem of "integration", i.e. of understanding how the body can glue the infinitesimal thermodynamical systems to obtain the global one. Owen proposes to approach this problem through the notion of sheaf, that is, by studying a certain kind of functor from a category of parts of the body into a category of abstract (or topological,or bornological) thermodynamical state-and-process systems. I am convinced that this line of thought will become important. I will comment separately on both the category of parts and the category of systems, with special reference to the utility of considering the systems themselves as categories.

The theory of parts of a body, discussed as a necessary preliminary in the articles of Noll and Williams, naturally concentrates on the subbodies (which might with luck form a Boolean algebra) but also must take account of boundaries (which are not sub-bodies). A convenient algebraic structure which includes both these features is that of a cartesian-closed partially-ordered set in which " $\longrightarrow$ " is thought of as " $\supseteq$ " and hence "cartesian product" becomes $\cup$ while "exponentiation" becomes a binary operation akin to subtraction, which is characterized by

$$A \supseteq C\backslash B \quad \text{iff} \quad A \cup B \supseteq C$$

The resulting algebraic system may alternatively be described as a lattice admitting such a subtraction; the subtraction is unique if it exists and its existence implies distributivity of the lattice operations. The system of all closed subsets of a given toplogical space is a typical example to keep in mind; the subtraction operation in that case is forced to be the closure of the set-theoretic difference. However, examples not of that form arise in various parts of mathematics, and it might reasonably be hoped that models of bodies which involve entities more (or less!) sophisticated than closed sets would still admit the structure here discussed. If 1 denotes the whole body, then we can define

$$\sim A = 1\backslash A$$

as a special case of the subtraction; $\sim A$ is thus characterized as the smallest object in the system for which $\sim A \cup A = 1$. One always has $\sim(A \cap B) = \sim A \cup \sim B$, but it can happen that $\sim(A \cup B) \neq \sim A \cap \sim B$; however $\sim\sim(A \cup B) = \sim\sim A \cup \sim\sim B$ always holds. Then

$$\sim\sim A \subseteq A$$

is the <u>regular</u> core of A, so that, with Noll and Williams (and Tarski), we may consider a part A of 1 to be a sub-body, or simply a body, if and only if $\sim\sim A = A$. But in such a lattice we may also define

$$\partial A = A \cap \sim A$$

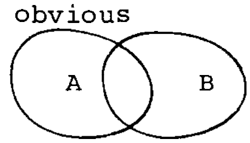
and consider this as the boundary of A in the sense of the system considered. That the notion of boundary is just that of "logical contradiction" (within the realm of closed sets) follows at once from the intuitive notion of motion: indeed, since the unit interval is connected, any continuous path which is in A at time 0 and in $\sim A$ at time 1 must at some intermediate time be in both A and $\sim A$, i.e. must pass through the boundary of A. Independently of the motivating example of closed parts, a great many useful identities can be proved in general for any lattice which satisfies our axiom of subtraction. For example

$$A = \sim\sim A \cup \partial A$$

for all A, and (as pointed out by R. Flagg)

$$\partial (A \cap B) = ((\partial A) \cap B) \cup (A \cap \partial B)$$

for all A,B. The latter "Leibniz formula" is remarkable in that, although it is pictorially obvious



and easily proved algebraically from our axioms (or in particular from the definition of closed set), and although similar formulas are well-known for more sophisticated objects such as currents, etc. the fact that it is true for ordinary boundaries of ordinary closed sets seems to have escaped the authors of textbooks on general topology; indeed the only source we could find for it (for which I thank G. Rousseau) is a little-known article by M. Zarycki (Fund. Math. 1927). Also evident from the above picture, and valid in any system of the kind under discussion, is

$$\partial (A \cup B) \cup \partial (A \cap B) = \partial A \cup \partial B$$

so that in particular

$$A \cup B = 1 \Longrightarrow \partial (A \cap B) = \partial A \cup \partial B$$

$$(\partial A) \cap B = 0 = A \cap \partial B \Longrightarrow \partial (A \cup B) = \partial A \cup \partial B$$

The special parts which are boundaries form an ideal (of "nowhere dense" parts), whose elements can also be characterized by many alternative equations, for example

$$\partial A = A, \text{ or } \sim A = 1, \text{ or } \sim\sim A = 0, \text{ or } A \cup Y \subseteq \sim A \cup Y$$

for all Y. Note that $\partial\sim A = \partial\sim\sim A$ is in general smaller than ∂A; equality of all three can hold only if A is a body.

A crucial relation between sub-bodies is that they be separate, or equivalently that they be (at most) in contact. For $\sim\sim A = A$, $\sim\sim B = B$, the following possible definitions of this relation are equivalent:

$$\sim B \supseteq A$$
$$\sim A \supseteq B$$
$$\sim\sim(A \cap B) = 0$$
$$A \cap B = \partial A \cap \partial B$$

By an interaction is meant a function H (in general vector valued) defined for separate pairs of bodies and such that $H(A,-)$ is additive on separate pairs and likewise $H(-,B)$. The importance of the material point of view having been recognized, it would be desirable to analyze, insofar as possible without reference to the momentary embeddings of the body in space, properties of interactions such as the property of being a surface interaction. By a surface interaction I here mean an interaction H such that whenever A_1, A_2 are bodies separate from the body B

$$A_1 \cap B = A_2 \cap B \Longrightarrow H(A_1, B) = H(A_2, B)$$

and similarly in the other variable (the last being automatic in case $H(A,B) = -H(B,A)$ holds.) This turns out to be equivalent to the vanishing of H on certain pairs. Namely, call a pair D,B very separate (or in at most slight contact) if

$$\sim(D \cup B) \supseteq D \cap B$$

or equivalently both

$$\partial(D \cup B) = \partial D \cup \partial B \quad \text{and} \quad \partial(D \cap B) = \partial D \cap \partial B.$$

Then an interaction H is a surface interaction iff $H(D,B) = 0$ whenever D,B are very separate. The crucial constructions for the proof of the foregoing statement are the definition of two "differences"

$$D_i = \sim\sim((A_1 \cup A_2) \cap (\sim A_i))$$

which will be very separate from B whenever A_1, A_2 are bodies having equal contact with B , and conversely, the definition of a special pair

$$A_1 = \sim(D \cup B)$$
$$A_2 = A_1 \cup D$$

which will have equal contact with B whenever D is a body very separate from B . It seems that still more results could be obtained within the intrinsic "body" point of view, without involving absolute continuity with respect to a surface measure which depends on the instantaneous embeddings of the body in space.

A system of states and processes can also be usefully construed as a small category $\underline{X}$, in which domain and codomain are simply the beginning and end states of a given process and composition is simply the operation of following one process by another. A category of this sort is often equipped with a "duration" functor to the additive monoid of nonnegative time translations; the functoriality is just the equation

$$\text{dur}(\beta\alpha) = \text{dur}(\beta) + \text{dur}(\alpha)$$

for composable processes, and

$$\text{dur}(1_X) = 0$$

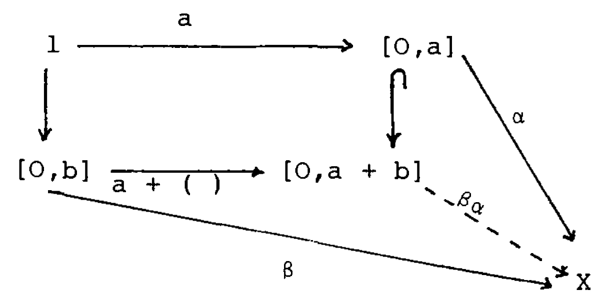
for the identity process of any state x . In examples the functor dur satisfies a further axiom of "unique lifting of factorizations" as follows

$$\text{dur}(\gamma) = s + t \Longrightarrow \exists!\ \alpha, \beta\ [\beta\alpha = \gamma,\ \text{dur}(\beta) = s,\ \text{dur}(\alpha) = t]$$

When the latter condition holds, the state with which α ends could be considered as the state " $\gamma(t)$" through which γ passes at the intermediate time t . Thus each morphism (process) γ in such a category $\underline{X}$ determines a path through the objects (states), and indeed a large class of examples can be constructed by starting with an object X in a topological category $\mathcal{X}$, defining the objects of $\underline{X}$ to be the points of X and defining the morphisms of $\underline{X}$ to be arbitrary continuous paths in X whose domains are intervals [0,a] . Again the presumption that $\mathcal{X}$ is a topos assists in constructing the totality of such paths as a single object (if needed):

$$\begin{array}{ccc} P(X) & \longrightarrow & \tilde{X}^R \\ \downarrow & \text{pullback} & \downarrow \text{dom}^R \\ R^+ & \longrightarrow & \tilde{1}^R \end{array}$$

where $\tilde{X}$ is the partial-map classifier which exists in any topos, $\tilde{1}^R$ is the power "set" of R, and the lower map is the one which assigns $[0,a]$ to any a. The fact that $PX \rightrightarrows X$ is a category follows from the pushout property of intervals in the <u>topological</u> topos:



It is essentially the failure of this pushout property (i.e. of the failure of paths to be closed with respect to the indicated composition) in a smooth (as opposed to continuous) topos which forced historically the introduction of <u>piecewise</u> smooth paths; the possibility of collecting even these into one "smooth" object is one of the important ingredients in the work of Chen. A general explanation for this possibility is the following: the <u>subdivided</u> smooth paths can in any case be collected into a "simplicial" object X^* in $\mathcal{X}$ and the inclusion $\mathrm{Cat}(\mathcal{X}) \hookrightarrow \mathcal{X}^{\Delta^{op}}$ has a left adjoint (for any reasonable $\mathcal{X}$), which "completes" a simplicial object to a category. Physically more typical examples of such categories $\underline{X}$ are obtained as non-full <u>sub</u>categories wherein one restricts attention to "admissible" processes obeying some constitutive relation. More precisely, there is often a functor

$$\begin{array}{c} \underline{X} \\ \downarrow \pi \\ \underline{C} \end{array}$$

from states to "configurations" which satisfies determinism in the form of the categorical <u>fibration</u> condition: Given any object (state) x in $\underline{X}$ and any morphism (deformation process) γ in $\underline{C}$ such that γ starts at $\pi(x)$, there is a unique morphism (state process) $\bar{\gamma}$ in $\underline{X}$ for which $\pi(\bar{\gamma}) = \gamma$ and $\bar{\gamma}$ starts at x. Under this condition clearly $\underline{X}$ is not usually a full category of paths even if $\underline{C}$ may be. Note that following such a fibration π with a duration functor for $\underline{C}$ will provide a duration functor for $\underline{X}$. For a given category $\underline{C}$ equipped with a duration there is a canonical example [Noll] of such $\underline{X}$ (of importance in the theory of materials with fading memory) in which the objects of $\underline{X}$ are taken to be <u>histories</u> in $\underline{C}$. Here a history is any functor $[0,\infty)^{op} \xrightarrow{x} \underline{C}$ (from the reversed <u>ordered</u>

set of nonnegative time translations) for which $\mathrm{dur}\circ x = \Delta$, where Δ is the "difference" functor from the ordered set of time translations to the monoid of time translations. Morphisms between histories are just arbitrary natural transformations of such functors, and a functor π can be defined as the restriction (along the indicated full inclusion) of the "evaluation at 0" functor

$$\mathrm{Hist}(\underline{C}, \mathrm{dur}) \hookrightarrow \underline{C}^{[0,\infty)^{op}} \xrightarrow{\ \mathrm{ev}\ } \underline{C}, \qquad \pi : \mathrm{Hist}(\underline{C}, \mathrm{dur}) \to \underline{C}$$

That π satisfies the above fibration condition follows from the "unique lifting of factorizations" property assumed for the duration functor on $\underline{C}$, using the fact that the time-translation-monoid is 1) commutative and 2) cancellative; for then 1) given any $\cdot \xrightarrow{\zeta} \cdot \xrightarrow{\gamma} \cdot$ in $\underline{C}$ there are unique γ', ζ' in $\underline{C}$ such that

$$\begin{array}{ccc} \cdot & \overset{\gamma'}{\dashrightarrow} & \cdot \\ {\scriptstyle\zeta}\downarrow & & \vdots\,{\scriptstyle\zeta'} \\ \cdot & \underset{\gamma}{\longrightarrow} & \cdot \end{array} \qquad \text{is commutative}$$

and $\mathrm{dur}(\gamma') = \mathrm{dur}(\gamma)$
$\mathrm{dur}(\zeta') = \mathrm{dur}(\zeta)$

and 2) every morphism in $\underline{C}$ is a monomorphism. (Note however that no non-identity morphism in $\underline{C}$ is invertible since 3) $t + s = 0 \Longrightarrow t = 0$ and $s = 0$). There are endomorphisms of non-zero duration which are nonetheless constant as paths - such freezes in a configuration category $\underline{C}$ can have non-trivial dynamical consequences in a state category $\underline{X}$. While when working in a single category it is natural to identify the concept of "cyclic process" with the concept of endomorphism, when comparing two categories by a functor such as $\pi : \underline{X} \longrightarrow \underline{C}$, there is the important phenomenon of hysteresis which must be kept in mind: a morphism $\overline{\gamma}$ in $\underline{X}$ may be such that $\gamma = \pi(\overline{\gamma})$ is cyclic in $\underline{C}$ even though $\overline{\gamma}$ itself is not cyclic in $\underline{X}$. Finally we point out that in many cases it suffices to consider the categories $\underline{C}$ as abstract categories, that is as defined in the topos $\mathcal{S}$ of abstract sets rather than in some higher topological topos $\mathcal{X}$; for a notion of closed set A of configurations (or states) can be derived from the category structure together with the duration functor by requiring that for every γ , $\gamma^{-1}(A)$ is closed in R , and this will often agree with the original topology.

An important way in which the fibration property for a functor π may arise is from a uniqueness and existence theorem for a non-autonomous ordinary differential equation, in which the paths in the lower category $\underline{C}$ represent the time variation of the equation itself, whereas the paths in the upper category $\underline{X}$ are actual solutions. It seems in some contexts needlessly confusing to combine a discussion of this relationship with a discussion of the separate question of the extent to which the paths in $\underline{C}$ themselves are the integrals of their derivatives.

An important role for state categories $\underline{X}$ in thermodynamical theory is to act as domains for quantitative "supplies" (or "actions") $\mathscr{S}$ which are simply functors from $\underline{X}$ into the additive monoid of a linear space or the extended reals. The "additivity" on paths expressed by the functoriality

$$\mathscr{S}(1_x) = 0$$
$$\mathscr{S}(\beta\alpha) = \mathscr{S}(\beta) + \mathscr{S}(\alpha)$$

together with some reasonable continuity condition, suggests that the value of $\mathscr{S}$ at α should be an integral, as indeed it is in many examples; there seems to be a need for representation theorems which would clarify the extent to which a general functor in this context can be expressed by a generalized integral formula. But the functoriality itself, supplemented by assumptions of a qualitative nature, suffices for a great many conceptual results of importance.

When the values of a given supply functor $\mathscr{S}$ are extended reals, particular interest attaches to the possibility of an "entropy" function S of states alone which bounds $\mathscr{S}$ in the sense that

$$\mathscr{S}(\alpha) \leq S(x_2) - S(x_1) \quad \text{whenever} \quad x_1 \xrightarrow{\alpha} x_2 \quad \text{in} \quad \underline{X} .$$

With due attention to the subtleties of addition and subtraction of extended reals (essentially subtraction is <u>adjoint</u> to addition rather than in general inverse to it, so that $[-\infty,\infty]$ becomes a (non-cartesian) <u>closed</u> category, with respect to $\leq$) such an S obviously exists, namely

$$S(x) = \sup\left\{\mathscr{S}(\alpha) \;\middle|\; x_0 \xrightarrow{\alpha} x_1\right\}$$

If $\mathscr{S} < \infty$ and $S(x_0) < \infty$, then $S(x) < \infty$ for all the x for which there exists at least one process $x \dashrightarrow x_0$; the condition that $S(x_0) < \infty$ is equivalent to the Clausius property $S(x_0) = 0$.

Coleman and Owen showed in 1974 that under suitable conditions not only the above naively-defined S but even its upper-semi-continuous regularization $\overline{S}$ bounds $\mathscr{S}$; a formulation of their theorem in the simplified setting described above was recently obtained [Lawvere]. But for each of the diverse special classes of materials to which their theory applies, Coleman and Owen obtained much more, namely that $\overline{S}$ is partially differentiable and that certain equations follow (from the above inequality) which relate the derivatives of $\overline{S}$ to the thermomechanical constitutive relations of the materials. These more precise conclusions have not yet been incorporated into the general theory, but it can be hoped that the present volume contains some of the necessary components for such an advance.

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