CATEGORICAL DYNAMICS

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[The following is intended as a summary of some lectures which I gave at several places in 1967. In these lectures, I offered some preliminary calculations in support of a program to (3) axiomatize the foundations of continuum mechanics in the spirit of Walter Noll on the basis of (2) a direct axiomatization of the essence of differential topology using results and methods of the French work in algebraic geometry (some of which I had learned from Gabriel); but I further maintained that this requires (1) axiomatic study of categories of smooth sets, similar to the topos of Grothendieck, since the most natural form of (2) is incompatible with "usual" set theory. Now, since my joint work with Tierney in 1969-1970, several conferences, many articles, and even one published book (by Johnstone) have been devoted to carrying out part (1) of this program. Meanwhile, a serious start on part (2) by Wraith and Kock has been followed by several further contributions, and in particular Dubuc in August 1978 explicitly demonstrated the consistency of part (2) by constructing a category in which ordinary differential topology is fully embedded but which moreover, satisfies the set-theoretically outragious axioms suggested by algebraic geometry. Work on (2) is far from complete (for example, it now seems that an approach in this spirit to differential forms involves still further divergence from "usual" set-theoretical logic). However, the growth of confidence in the program engendered by these developments has also led to a growth of interest in the origin of the program itself. I am taking advantage of this current interest to publish this summary, along with the observation that seriously taking up part (3) of the program will surely lead in particular to further illumination of parts

(1) and (2). Of course, the framework of "ordinary" set-theory has not succeeded to prevent Noll's own work from advancing; two fundamental works from the early 1970's are included in his selected papers published by Springer. My main external sources for the following summary have been page 937 of volume 14 of the Notices of the AMS and especially notes taken by Saunders MacLane on May 19, 1967 at Chicago and on November 25, 1967 at Urbana, which he very kindly sent to me in summer 1978. Some remarks based on more recent developments have been inserted into the summary between brackets [ ].]

I hope that categorical methods can be used to give a simple axiomatic basis for parts of mathematics which arose from physics (particle mechanics, fluid mechanics, differential geometry, harmonic analysis, etc). Some physicists and engineers seem in effect to have the insight that geometrical and physical constructions can be performed, with almost as much freedom as sets can be defined in naive set theory, without ever leaving the realm of smooth objects and smooth maps. But usual mathematical models, such as the category of smooth manifolds, on the one hand presuppose a long intricate purely mathematical construction (there does not seem to be an intrinsic description of that category which could reasonably be taken as a "simple" starting point) and on the other hand are poor in regard to closure properties since even something so fundamental (for calculus of variations etc) as the smooth space of smooth maps between two smooth spaces is ambiguous and difficult, and pullbacks in general don't exist.

As I emphasized in my 1971 - 72 Aarhus lectures, not only the function space but also the smooth space of smooth subspaces and the smooth space of representations of a given smooth group "should" have clear meanings according to such insight. But rather than scoffing at insight (which some seemed to have considered the only healthy public response in recent decades) we can try to axiomatically express what some aspects of it might mean precisely and also to construct mathematically acceptable models of such axioms, in the hope ultimately of actually clarifying the learning, development, and use of these branches of mathematics. From 1966 Oberwolfach lectures by M. Demazure and P. Gabriel I learned some facts and methods which seem important both for the axiomatics and for the construction of models, essentially the Cartier-Grothendieck functorial approach to algebraic groups since published in Springer Lecture Notes # 151 (1970) and a 1969 North-Holland book by Demazure-Gabriel.

Consider a category $\mathcal{X}$ in which we have a given ring object R. About $\mathcal{X}$ we will assume that it has a terminal object 1, pullbacks, and for each $X \xrightarrow{f} Y$, a right adjoint

$$(1) \qquad \mathcal{X}/X \xrightarrow[\Pi_f]{} \mathcal{X}/Y$$

to the functor f^* of pulling back along f. This implies that each $\mathcal{X}/X$ has an internal hom right adjoint to product over X, denoted by exponentiation. Thus $\mathcal{X}$ is what came to be called, after the work of Penon, a locally cartesian-closed category.

(Later we will need one construction which is most easily guaranteed by assuming $\mathcal{X}$ has countable coproducts and coequalizers.) Objects of $\mathcal{X}$ are to be thought of as smooth spaces, and morphisms $X \longrightarrow R$ are to be thought of as quantities smoothly varying over X. Note that for example $Hom_R(A,B)$ for two R-modules has a well-defined meaning as a subobject of B^A. R-modules are to be thought of as vector spaces (with a smooth structure) even though we do <u>not</u> assume R is a field. The geometric origin of R is roughly as follows. In $\mathcal{X}$ there are Euclidean spaces E_1, E_2, E_3 whose structure (=basic geometric constructions) are given by morphisms of $\mathcal{X}$. In particular there are <u>abelian</u> subgroups

$$V_n = \text{Trans}\ (E_n) \subset E_n^{\ E_n}$$

of translations and hence rings

$$R_n = \text{Hom}\ (V_n,\ V_n).$$

$R = R_1$ is <u>commutative</u> because of two facts: E_1 is <u>one</u> - dimensional, and every homomorphism $V_1 \longrightarrow V_1$ is a homothety because it, like every map in $\mathcal{X}$, is smooth. Of course from analytic geometry we know essentially how to use cartesian products to construct co-ordinatized models of E_n, imagining in inverted fashion that we start with the datum R.

The second axiom will permit an intrinsic theory of differentiation to be developed. We assume given a subobject $D \subset R$ which contains the zero quantity $1 \xrightarrow{0} R$ and which is to be thought of as the space of first-order infinitesimal quantities.

For any object X, the object X^D will be thought of as the <u>tangent bundle</u> of X, with projection $X^D \longrightarrow X$ induced by $1 \xrightarrow{0} D$, and for any morphism $X \xrightarrow{f} Y$ in $\mathcal{X}_0$, f^D will be thought of as the derivative of f. Thus a tangent vector $D \longrightarrow X$ to X is at the point $1 \xrightarrow{0} D \longrightarrow X$ of X, and the derivative f^D of f takes any tangent vector $D \xrightarrow{v} X$ at a point x to the tangent vector $D \xrightarrow{v} X \xrightarrow{f} Y$ at the point fx. The functoriality of exponentiation $(\ )^D$ is thus essentially the chain rule for differentiation. To prove 1) the Leibniz rule (for differentiation of variable quantities) as well as that 2) there are precisely the right amount of tangent vectors for R and related spaces, we assume our second axiom(2): We need that D is closed under the action of the multiplicative monoid R, and that the composites

$$\text{(a)} \qquad D \rightarrowtail R \underset{(\ )^2}{\overset{0}{\rightrightarrows}} R$$

are equal, where $(\ )^2$ denotes the squaring map from the ring structure of R and 0 denotes the constantly 0 map $R \longrightarrow 1 \xrightarrow{0} R$, and we also need that there is an isomorphism

$$\text{(b)} \qquad R^D \cong R \times R$$

In fact, we may as well define D by requiring that (a) be an equalizer, and assume (b). [* However, as Massimo Galuzzi and Gian-Carlo Meloni calculated in July 1978, (a) follows from (b) if we assume that $\frac{1}{2} \in R$ and interpret (b), as in the meantime had been done in several papers by Anders Kock, to mean that the canonical morphism $R^D \longleftarrow R \times R$ is invertible *]. Though there are many morphisms $R \longrightarrow R$ (there are at least all the polynomials), upon restricting to D they all become <u>linear</u>; but on the other hand D is large enough so that distinct linear (i.e. affine) maps $R \rightrightarrows R$ have distinct restrictions $D \rightrightarrows R$. We need category theory for this axiom, since it seems no such ring could exist in classical set theory [as was proved in considerable generality by calculations in the mid-70's by Kock, Schanuel, and Lawvere]. The condition (b) is not restricted to "line-like" R, since it follows that for any R-module V

$$(V^*)^D = V^* \times V^*$$

canonically, since

$$\mathrm{Hom}_R(V,R)^D = \mathrm{Hom}_R(V,R^D) = \mathrm{Hom}_R(V,R\times R) = V^* \times V^*$$

However, many vector spaces are not dual modules and it is less clear how to compute their tangent bundles. But it is trivial that for any X, Y in $\mathcal{X}$

$$(Y^X)^D = (Y^D)^X$$

showing how "easy" the smooth structure ofinfinite-dimensional objects really is. Using (b) we can define the pre-gradient of any variable quantity $X \xrightarrow{f} R$ to be the composite

$$X^D \xrightarrow{f^D} R^D \cong R \times R \xrightarrow{\pi'} R$$

where π' is the other projection, the one not corresponding to the map induced by $1 \xrightarrow{0} D$. Also the interpretation of tangent vectors as distributions ("of compact support") is given by the morphism

$$X^D \longrightarrow \mathrm{Hom}_R(R^X, R)$$

corresponding to

$$X^D \times R^X \longrightarrow R^D \xrightarrow{\pi} R.$$

Note that differentiation is itself a smooth map

$$Y^X \longrightarrow (Y^D)^{(X^D)}$$

as is the pre-gradient

$$R^X \longrightarrow \mathrm{Hom}^{\bullet}_R(X^D, R)$$

where $\mathrm{Hom}^{\bullet}_R$ denotes morphisms which are homogeneous with respect to the action of the <u>multiplicative</u> <u>monoid</u> R. This monoid acts on D, hence on X^D. On the other hand, <u>additon</u> of tangent vectors

$$X^D \times_X X^D \xrightarrow{+} X^D$$

only exists under the assumption on X, that the functor $X^{()}$ takes certain non-pushout squares of D-like objects into products in $\mathcal{X}/X$, [essentially what is called "condition E" in SGA3 as I noticed in April 1979]. On the other hand, since C^∞ maps which are everywhere defined on a vector space and homogeneous of degree 1 are automatically additive, we may expect that

$$\mathrm{Hom}_R(V,V') \hookrightarrow \mathrm{Hom}^{\bullet}_R(V,V')$$

has a strong tendency to be an isomorphism in our $\mathcal{X}$, and that in particular

$$\mathrm{Hom}^{\bullet}_R(R^D, R)$$

may serve as a reasonable surrogate for

$$\mathrm{Hom}_{R \times X}(X^D, R \times X).$$

even when X^D is not additive over X.(Here we imagine that Hom has been given some rational definition using the rich supply of additive relations induced by those not-necessarily-pullback squares of multiple tangents over X).

[(After reading Kock's exposés on Synthetic Differential Geometry from the Benabou Seminar Jan. 1979) The natural extension of the axiom (b) itself to multiple and higher tangents seems to be to consider the category $\mathcal{W}$ of all commutative R-algebras W in $\mathcal{X}$ with the following properties

$$W = R \oplus H$$

$$H \cong R^k \text{ as R-modules, some } k \in N$$

$$H \underset{(\;)^{p+1}}{\overset{0}{\rightrightarrows}} H \text{ equal for some } p \in N$$

and to define

$$D(W) = \underline{\mathrm{Alg}}_R(W,R).$$

and then require that the natural map

$$W \longrightarrow R^{D(W)}$$

into the double dual be an isomorphism for all W in $\mathcal{W}$. This implies again the same statement for any dual vector space

$$\begin{array}{ccc} V^* \otimes_R W & \xrightarrow{\approx} & (V^*)^{D(W)} \\ & \searrow^{\cong} \quad \nearrow^{\cong} & \\ & \mathrm{Hom}_R(V,W) & \end{array}$$

]

We define a <u>vector field</u> on an object X in $\mathcal{X}$ to be any section v of $X^D \longrightarrow X$, and a <u>morphism of vector fields</u> $X,v \longrightarrow X',v'$ to be any f in $\mathcal{X}$ such that

$$\begin{array}{ccc} X^D & \xrightarrow{f^D} & X'^D \\ \uparrow v & & \uparrow v' \\ X & \xrightarrow[f]{} & X' \end{array}$$

is commutative. We thus get a category $Vect_R(\mathcal{X})$. Because our tangent concept is representable by a single generic object D, the notion of vector field can be equivalently expressed in the simpler form

$$\begin{array}{ccc} X \times D & \xrightarrow{\bar{v}} & X \\ {\scriptstyle 0}\nwarrow & & \nearrow{\scriptstyle 1_X} \\ & X & \end{array}$$

with a corresponding form of the notion of morphism

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \uparrow \bar{v} & & \uparrow \bar{v}' \\ X \times D & \xrightarrow[f \times D]{} & X' \times D \end{array}$$

When convenient, the notion of vector field can be equivalently expressed in a third way:

$$\begin{array}{ccc} D & \xrightarrow{\bar{\bar{v}}} & X^X \\ {\scriptstyle 0}\nwarrow & & \nearrow{\scriptstyle \ulcorner 1_X \urcorner} \\ & 1 & \end{array}$$

The object R carries a canonical vector field (essentially the derivative of the identity) so that for any path $R \longrightarrow X$, its derivative can be composed with it to yield a path of tangent vectors.

The notion of vector field is usually taken to be the basic notion of "differentiable dynamical system", in infinitesimal form. The corresponding integrated form, is a flow or action of the additive group R; in the continuous case the study of such is called "topological dynamics". The narrow meaning of the term "categorical dynamics" is thus analogous to the use of "cat" as a variable which can take values like cat=top, cat = diff, cat = PL, etc., i.e. the study of $\mathcal{X}$-flows, where $\mathcal{X}$ denotes a pair $\mathcal{X}$,R satisfying our two axioms and where a flow is a pair X, $X \times R \longrightarrow X$ in $\mathcal{X}$ satisfying the usual axioms

$$xe^{0} = x$$

$$xe^{t_1+t_2} = (xe^{t_1})e^{t_2}$$

where this use of the symbol e is solely for notational harmony. A morphism f of flows satisfies

$$f(xe^{t}) = (fx)e^{t}.$$

Thus we have a category $\mathrm{Flow}_R\ (\mathcal{X})$ of $\mathcal{X}$-dynamical objects.

Now since $D \subset R$, every flow $X \times R \longrightarrow X$ restricts to a vector field $X \times D \longrightarrow X$ by considering only those time-lapses infinitesimally close to 0, yielding a functor

$$\mathrm{Flow}_R\ (\mathcal{X}) \xrightarrow{(\ )^{\cdot}} \mathrm{Vect}_R(\mathcal{X})$$

which preserves underlying space. The problem of integrating a system of ordinary differential equations could thus be viewed as having two parts, namely applying an adjoint to the functor $(\ )^{\cdot}$ and then studying to what extent the underlying space has been changed by such "integration". Actually the above functor has two adjoints, which might fancifully be called the "upper and lower integrals of a vector field". The right adjoint can be seen to exist without further ado as

$$\mathrm{Hom}_D(R,X)$$

the subspace (of the space of all complete paths $R \longrightarrow X$) consisting of morphisms from the canonical vector field on R to the given one v on X, or briefly the subspace consisting of solution curves for the infinitesimal flow v. This solution space carries a natural flow, induced by translations on R itself, whose corresponding infinitesimal flow is mapped morphically back to X,v by the evaluation at 0

$$\begin{array}{c} \mathrm{Hom}_D(R,X) \\ \downarrow \in \\ X \end{array}$$

which to every solution curve assigns its underlying initial-value at time 0. The properties of injectivity or surjectivity of $\in$ express exactly the uniqueness or existence theorem for the initial-value problem for the ODE system determined by v.

To calculate the left adjoint

$$\mathrm{Vect}_R\,(\mathcal{X}) \xrightarrow[(\ )\otimes_D R]{} \mathrm{Flow}_R\,(\mathcal{X})$$

to $(\ )^{\bullet}$, we need the existence of coequalizers

$$X \times D \times R \rightrightarrows X \times R \longrightarrow X \otimes_D R$$

$$x, h, t \rightsquigarrow x, t+h$$
$$\rightsquigarrow xe^h, t$$

where we have written $xe^h = v(x,h)$ and where of course R acts on $X \otimes_D R$ by $\langle x,s\rangle e^t = \langle x, s+t\rangle$. To compute $X \otimes_D R$ in a particular case is more difficult, as it depends not only on detailed knowledge of v but also on detailed knowledge of coequalizers in $\mathcal{X}_0$. An approximation to such a computation may be useful along the following lines. Let $\mathcal{X}_0$, R be as appropriate for algebraic or analytic geometry over a field of characteristic 0 (see below) and suppose X = spec(A) for a commutative algebra A. Then a vector field on X can be identified with a derivation d_v (Leibniz rule) on A since elements of A are identified with morphisms $X \xrightarrow{f} R$ and we can always form

$$\underbrace{X \xrightarrow{v} X^D \xrightarrow{f^D} R^D \cong R \times R \xrightarrow{\pi} R}_{d_v f}$$

Now define (in sets $\mathcal{S}$)

$$A_v = \left\{ f \in A \;\middle|\; \exists n \geq 0 \;\left[d_v^{n+1}(f) = 0 \right] \right\}$$

in terms of iterates of d_v, a subalgebra of A, for which a flow on spec (A_v) can be explicitly defined by

$$\begin{array}{ccc} A_v & \longrightarrow & A_v[t] \\ f & \rightsquigarrow & \sum \dfrac{d_v^n(f)}{n\,!} \; t^n \end{array}$$

Then there is a unique morphism of flows such that the following diagram of morphisms of vector fields commutes

$$\begin{array}{ccc} X & \longrightarrow & X \underset{D}{\otimes} R \\ & \searrow & \vdots \downarrow \\ & & \text{spec}\,(A_v) \end{array}$$

Note that A_v is filtered into quantities invariant under the flow "on X", quantities whose time dependence is linear along the flow, etc. The value of such approximation seems limited, however.

If we also have countable coproducts in $\mathcal{X}$, then the two "integrals" for $(\ )^{\bullet}$ can be viewed as special cases of the very general adjoints $\mathrm{Hom}_S(R,-)$, $(\ )\otimes_S R$ associated to a homomorphism $S \longrightarrow R$ of any two monoids in $\mathcal{X}$. For we can define

$$e^D = \sum_{n=0} D^n/n!$$

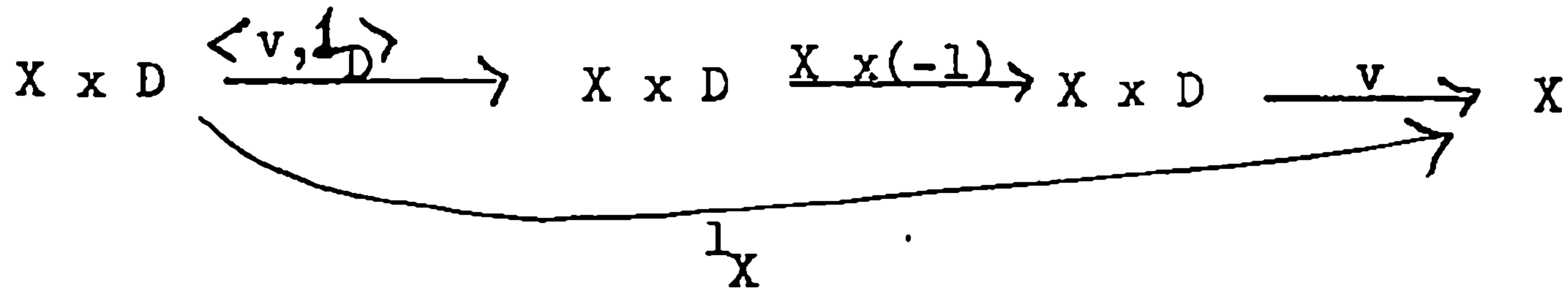
the free commutative monoid on the object D, where $(\ )^n/n!$ denotes the orbit space for the natural action of the symmetric group, and find a natural homomorphism $e^D \longrightarrow R$ induced by the inclusion $D \subset R$, and whose image is the ideal of R generated by D, consisting of all sums of elements of square 0. If S is defined as the quotient of e^D modulo the congruence relation determined by the condition that $1 \xrightarrow{0} D$ be congruent to the neutral element of e^D, then $e^D \longrightarrow S \longrightarrow R$ and we have

$$\mathrm{Flow}_R(\mathcal{X}) = \mathcal{X}^R \xrightarrow{(\ )^{\circ}} \mathcal{X}^S \cong \mathrm{Vect}_R(\mathcal{X}).$$

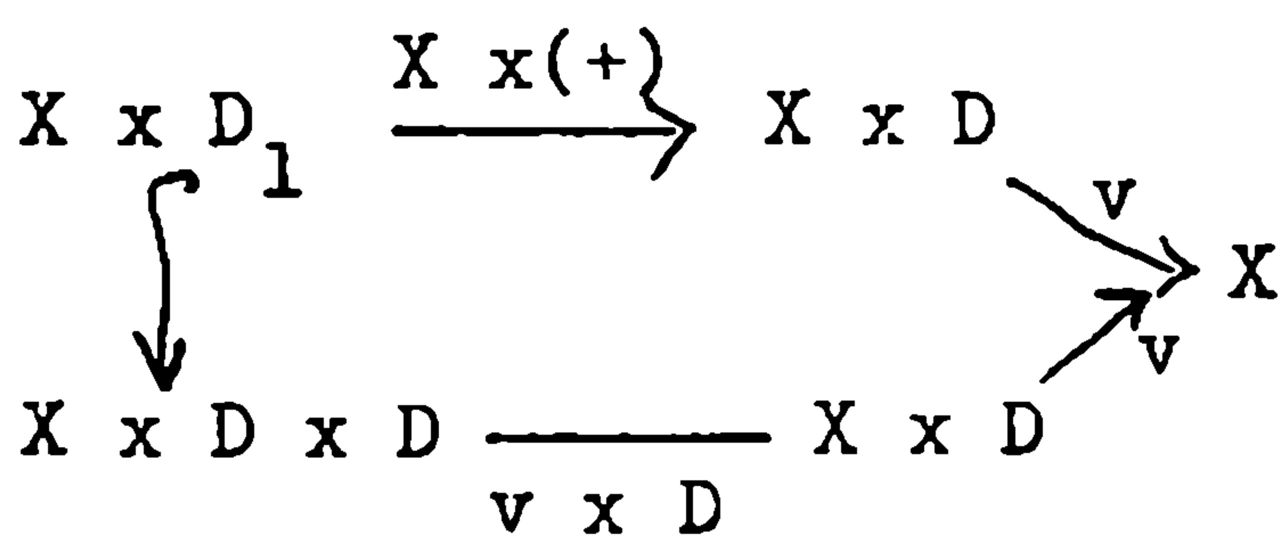
<u>provided</u> D is so small that every vector field in $\mathcal{X}$ also satisfies <u>infinitesimal commutativity</u>

$$\begin{array}{ccccc} X \times D \times D & \xrightarrow{v \times D} & X \times D & \overset{v}{\searrow} & \\ X \times \tau \updownarrow & & & & X \\ X \times D \times D & \xrightarrow[v \times D]{} & X \times D & \underset{v}{\nearrow} & \end{array}$$

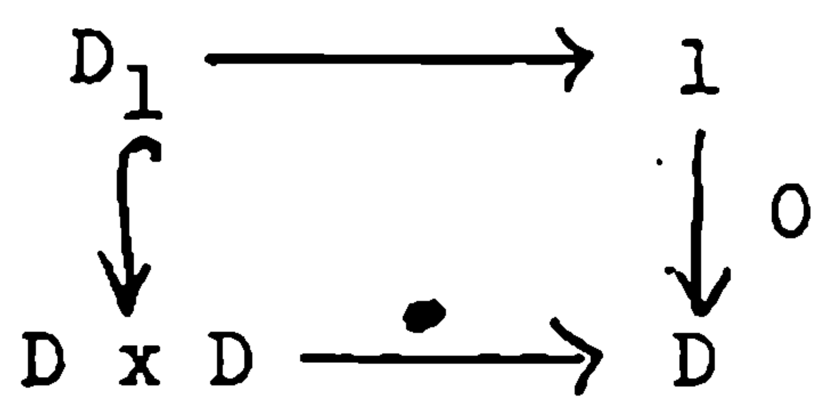
However, if it turns out that the latter is a special condition on X and/or v (i.e. does not follow from the <u>identity</u> axiom for vector-fields by itself) then one should consider its companion condition of <u>infinitesimal invertibility</u>



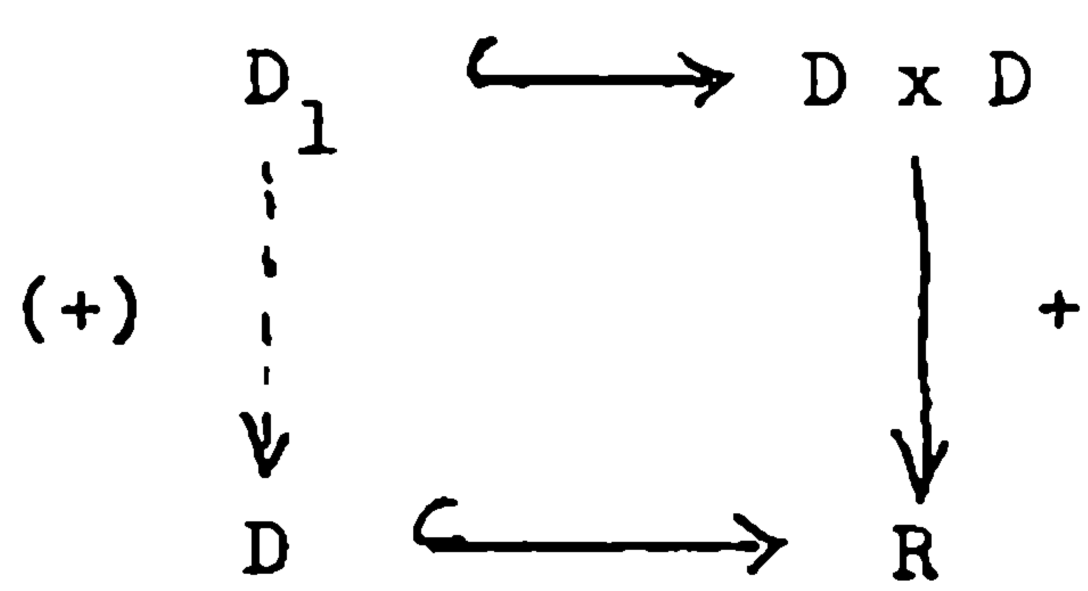
and even consider strengthening the concept of vector field by adding the following still stronger axiom of <u>infinitesimal associativity</u> to the definition:



Here D_1 is the infinitesimal neighborhood of the diagonal in D x D defined by the pullback with multiplication



whose importance here is that it is equally well defined by the pullback.



since we assume $\frac{1}{2} \in R$. (Even if the above additional laws are added to the definition of S, it is not clear whether $S \longrightarrow R$ becomes monic).

Vector fields are just the simplest kind of models of a differential-algebraic theory, where the latter refers to a concept more general than algebraic theories in $\mathcal{X}$, whose arities are natural numbers, but significantly less general than general monads (= triples) in $\mathcal{X}$, whose arities are arbitrary objects, namely we consider theories in $\mathcal{X}$ whose arities and co-arities are objects like D [i.e. more generally D(W) for W ∈ $\mathcal{W}$] where in general an operation of arity A and co-arity C on X means a map $C \times X^A \longrightarrow X$. The hope would be that more refined theorems as to coequalizers, etc. could be proved for such limited theories than could be true for arbitrary monads in X. Thus for example in ordinary algebraic theories we can deal with commutative algebras X with an additional unary operation f satisfying $f(x_1 + x_2) = f(x_1) \cdot f(x_2)$, but only with differential-algebraic theories as modeled in such $\mathcal{X}$ does it become "algebraic" to require also f' = f where ( )' is the <u>intrinsic</u> derivative for the underlying object of X (preceeded by $X \xrightarrow{\langle X, 1 \rangle} X \times X \cong X^D$ and followed by $X^D \cong X \times X \xrightarrow{\pi} X$). Even ordinary "abstract" algebraic theories, e.g. groups or Lie algebras, when extended naturally to "trivial" differential-algebraic theories, may have non-trivial morphisms of differential-algebraic theories between them.

As is well known, if G is a model of an algebraic theory in a category with exponentiation and if I is an object then G^I is a model of the same theory, and moreover maps $I' \longrightarrow I$ induce homomorphisms $G^I \longrightarrow G^{I'}$. For example if G is a monoid then the projection $G^D \longrightarrow G$ is a homomorphism of monoids (of groups if G is a group) and the kernel of that homomorphism is Lie(G). For example, $\text{Lie}(X^X) = \text{Vect}(X)$, the object whose elements are all the vector fields on X, which is thus seen to always carry an "addition" (maybe non-commutative even for the X which are so bad that addition on X^D in $\mathcal{X}/X$ does not exist; of course, if addition of tangent vectors does exist then the Eckmann-Hilton Lemma shows that the "addition" must be commutative since it must agree with addition by naturality. Since Lie is functorial for monoid homomorphisms, an associative action of G on a space X induces an "infinitesimal action"

$$X \times \mathrm{Lie}(G) \times D \longrightarrow X$$

of Lie(G) on vector fields on X. What is explicitly the monoid M obtained by dividing the free monoid generated by Lie(G) x D by all relations which are true in all actions induced from a global G-action? It is again clear in principle that there are left and right "integration" adjoints.

Now the functor

$$\mathrm{Gr}(\mathcal{X}) \xrightarrow{\mathrm{Lie}} \mathcal{L}ie\,(\mathcal{X})$$

is itself representable, in fact by the S previously discussed. However we don't know exactly what $\mathcal{L}ie(\mathcal{X})$ is; with respect to which doctrine of theories should the costructure of S be computed - partial differential-algebraic theories? [In the first circulated article following the synthetic approach suggested in the lectures here summarized, Gavin Wraith in the early 70's showed how the pullback conditions on multiple tangents of G needed to get the Lie-algebra structure on Lie(G) could be expressed and used in the axiomatic setting] For any definite interpretation of $\mathcal{L}ie(\mathcal{X})$ general principles say that Lie will have a <u>left adjoint</u>, and hence in particular for each $G \in \mathrm{Gr}(\mathcal{X})$ a co-adjunction homomorphism $\hat{G} \longrightarrow G$, whose kernel and cokernel are further definite groups which could be called $\pi_1(G)$ and $\pi_0(G)$....But whatever may be the complications which may lurk in "arbitrary" group objects, the above definitions and axioms are sufficient to calculate explicitly in $\mathcal{X}$ the Lie algebra of classical algebraic groups, e.g.

Lie(GL(n)) = R_n with commutator

Lie(SO(3)) = V_3 with cross product.

[See not only the writings of Demazure and Gabriel but also J.P. Serre's Benjamin book on Lie groups and Lie algebras. (1965)]

The physical study of a dynamical system involves not only a <u>state space</u> X equipped with a dynamical vector field, but actually a more specific construction of such in terms of simpler objects. Frequently there is a space Q of <u>configurations</u> and a given map $X \longrightarrow Q$ expressing that each state has an underlying configuration, but in general must involve more. For particle mechanic rigid body mechanics, and hydrodynamics one can define

$$X = Q^D$$

but this actually amounts to the very restrictive hypothesis that the <u>response</u> of the material depends only on the <u>infinitesimal history</u> of its motion, where motions are interpreted to mean paths $R \xrightarrow{q} Q$ in configuration space. In the "simple" cases just mentioned, the analysis of the required vector field on X is often associated with the study of a "Lagrangian"

$$\mathcal{L} : X \longrightarrow R$$

which induces a functional

$$Q^R \longrightarrow R^{R\times R}$$

called "action" by applying to

$$R \xrightarrow{\text{canon}} R \times R \cong R^D \xrightarrow{q^D} Q^D = X \xrightarrow{\mathcal{L}} R$$

(curved arrow from R to R labelled "$\mathcal{L}(q,\dot{q})$")

the integration process

$$R^R \xrightarrow{\int} R^{R \times R}$$

$$f \rightsquigarrow (\langle a,b\rangle \rightsquigarrow \int_a^b f(t)dt)$$

A possible (physically motivated) addition to our two axioms would be the existence of the morphism $\int$, but it is not clear what condition on it would be both desirable and possible. [one of the desirable ones would be

$$\int_a^{a+h} f(t)dt = f(a)h$$

for any h such that $h^2 = 0$. This would seem to yield an algebraic proof of the fundamental theorem of calculus, in conjunction with the additivity of $\int$ in each of its two kinds of argument.]

But more fundamentally, even if the rather abstract Lagrangian is useful, its construction and the construction of the vector field on states in a particular case involves the knowledge of <u>forces</u> and more particularly of an analysis of forces into three kinds; inertial, external, and internal mutual response. Such an analysis depends in turn on a more specific construction of the configuration space Q, which (even when X is more general than Q^D) is usually realized as a given subspace

$$Q \subseteq E^B$$

[of "placements"] where $E = E_3$ is the actual space and where B is the space of "particles" of the material body in question.

In particle mechanics, B is a finite discrete set, but in continuum mechanics it is usually a three-dimensional manifold [although in the theory of rods, cords, plates, and shells, B is perhaps a lower-dimensional object for which the fibers of $B^D \longrightarrow B$ are nonetheless three dimensional]. One of the motivations for the axiomatic theory of $\mathcal{X}$,R is to give simple expression to the old idea that the theory of the infinite-dimensional Q with dim(B)>0 should be in some respects "just like" the particle case [which was also a motivation for K.T. Chen's Urbana (1978?) notes on the calculus of variations, in which a category with some properties in common to our $\mathcal{X}$ is independently constructed].

A reasonable condition on

$$Q \subseteq E^B$$

would be that Q is mapped into itself by the induced action of the group of rigid motions of E. The group G(B,Q) of all those invertible endomorphisms of E which map Q into Q might thus serve as a crude measure of the distinction between very rigid bodies (G minimal) and rarefied gases (G maximal); however a more serious measure of the distinction of the kinds of material B is made of should involve infinitesimal symmetry of the internal mutual response functional, not discussed here.

When the simple definition of state space suffices, we have

$$X = Q^D \subseteq (E^B)^D = (E^D)^B = (E \times V)^B$$

where $E^D = E \times V$ with $V = V_3$, the translation vector space of the affine space $E = E_3$, and hence

$$X = Q^D \subseteq Q \times V^B$$

where V^B is the space of velocity fields on B. Inertial forces, momentum and kinetic energy involve not only velocity fields and a metric on V for their computation, but also a further given structure of a mass distribution on the body. Using the total mass of the body as unit so that the mass of parts can be measured in terms of pure quantities R, such distribution can be considered as an R-linear morphism

$$R^B \xrightarrow{m} R$$

which preserves constants and which is positive. {But what is the best way to account axiomatically for positivity? Do the elements of D support a notion of positivity or not? Unpublished lecture of André Joyal at Columbia University, December 1975 on "real algebraic geometry" gives some indications.} Integration with respect to m can then be applied to functions with values in convex sets such as E, yielding in particular a "center of mass" map

$$Q \longrightarrow E.$$

The mass distribution and the metric on E are the main ingredients in the analysis of one kind of external force and internal mutual response, namely gravitation. For more details on more subtle internal mutual response which material bodies may have, see papers of Walter Noll in the Archive of Rational Mechanics and Analysis, late 1950s {and especially Noll's Selected Papers published by Springer 1974} the main physical and mathematical ideas of which can hopefully be expressed in categories like our $\mathcal{X}$.

An important virtue of the categorical axiomatics we have indicated is that if there is one model $\mathcal{X}$ then there are immediately infinitely many other interesting and useful models for the whole theory, in fact at least two classes of such. If G is any group object in $\mathcal{X}$ (e.g. the Galilean or Lorentz group??) then the category $\mathcal{X}^G$ of G-actions and equivariant morphims is again a model for our axioms if we interpret R to mean R with trivial action. Also if M is any ("parameter") object in $\mathcal{X}$ then the category $\mathcal{X}/M$ of objects over M is also again a model for our axioms, interpreting R as R x M; theory of dynamical systems in $\mathcal{X}/M$ is the theory of families of dynamical systems in $\mathcal{X}$ parameterized by M [as in bifurcation theory, see Marsden BAMS vol. 84, Nov. 1978]. [It was, as briefly indicated in paper for the Eilenberg volume, qualitative and unpublished considerations of the kind just mentioned, as much or more than published problems of independence, etc. in abstract set theory and logic, which were an important impetus toward the 1969 -70 Lawvere-Tierney development of essentially algebraic axioms for topos theory.]

Now we consider three general categorical constructions which are useful in showing the existence of models for our axioms as well as for suggesting possible stronger axioms. All our models $\mathcal{X}$ are subgenerated by the algebraic theory $\underline{A}$ whose n-ary operations are by definition

$$\underline{A}(R^n,R) = \mathcal{X}(R^n,R)$$

where by "subgenerate" we mean (strongly) generated by the full subcategory $\underline{C}$ of $\mathcal{X}$ determined by those objects X which occur as equalizers

$$X \longrightarrow R^n \rightrightarrows R^m$$

Thus conversely we can construct such $\mathcal{X}$ by starting with a suitable algebraic theory $\underline{A}$ and considering the category $\underline{C}^{op}$ of finitely presented $\underline{A}$-algebras, i.e. those that occur as coequalizers of finitely generated free $\underline{A}$-algebras in the category $Alg(\underline{A}) = Lex(\underline{C},\mathcal{S})$ of $\underline{A}$-algebras. Then $\mathcal{X}$ is to be sought as a full subcategory of $\mathcal{S}^{\underline{C}^{op}}$ whose inclusion has a left-exact left adjoint, for then thanks to work of Giraud and Verdier in SGA4, we can conclude that the category $\mathcal{X}$, called a "topos" will satisfy our first axiom on the existence of $\prod$ and in fact have further useful exactness properties. As a matter of fact, the basic duality between algebra and geometry is just the restriction of an adjoint pair called "conjugacy" by Isbell:

$$\begin{array}{ccc}
(\mathcal{S}^{\underline{C}})^{op} & \rightleftarrows & \mathcal{S}^{\underline{C}^{op}} \\
\uparrow & & \uparrow \\
Lex(\underline{C},\mathcal{S})^{op} & & sh(\underline{C},\mathcal{S}) \\
\| & & \| \\
Alg(\underline{A})^{op} & \underset{\text{spec}}{\overset{\text{function algebra}}{\rightleftarrows}} & \mathcal{X} = \text{"Geom}(\underline{A})\text{"}
\end{array}$$

where both conjugates are defined by the same formula

$$\mathrm{conj}(\ )(C) = \mathrm{Nat}(-,C)$$

where the C on the right denotes the representable functor of the appropriate variance, and Nat refers to natural transformations of functors of the same variance. Thus

$$\text{(function algebra of } X)(C) = \mathcal{X}(X,C)$$

and

$$\mathrm{spec}(A)(C^{op}) = \mathrm{Alg}(\underline{A})(A,C^{op}).$$

Those $A \in \mathrm{Alg}(\underline{A})$ which are <u>inverse</u> limits of finitely-presented $\underline{A}$-algebras will satisfy

$$A = \text{function algebra of spec}(A).$$

There is still the choice of <u>which</u> subtopos $\mathcal{X}$ of $\mathcal{S}^{C^{op}}$ is more appropriate but note that the conditions "$\underline{C} \subseteq \mathcal{X}$" and "spec(A) $\in \mathcal{X}$ for all $A \in \mathrm{Alg}(\underline{A})$" are equivalent and provide a minimum restriction on this choice.

In order to satisfy the second axiom, we define

$$R \in \mathcal{X} \subseteq \mathcal{S}^{\underline{C}^{op}}$$

to be the underlying set functor on $\underline{C}^{op} \subseteq \mathrm{Alg}(\underline{A})$, or equivalently the functor represented by the free $\underline{A}$-algebra on one generator. Thus the spectrum of any algebra in $\underline{C}^{op}$ is a space form $C \in \underline{C}$ and any object $X \in \mathcal{X}$ is determined by a discrete fibration $\underline{C}/X$ over $\underline{C}$ whose fibers consist of all figures in X of a given form (possibly singular figures) and whose morphisms are "incidence relations" between such figures. But X also determines, by mapping into R, R^2, in general into all the equalizers $C \longrightarrow R^n \rightrightarrows R^m$ in $\mathcal{X}$, a discrete op-fibration $X/\underline{C}$ over $\underline{C}$ whose objects are all variable quantities on X satisfying various given equations, and whose morphisms are $\underline{A}$-algebraic operations on such quantities. The requirement that R be a commutative ring in $\mathcal{X}$ will be met if the theory $\underline{A}$ contains the algebraic theory of commutative rings as a subtheory, and then $D \in \underline{C}$ will be forced to be the spectrum of the $\underline{A}$-algebra obtained by dividing the free $\underline{A}$-algebra on one generator t by the $\underline{A}$-congruence relation generated by the one relation $t^2 = 0$, i.e.

$$D \in \mathcal{X} \subseteq \mathcal{S}^{\underline{C}^{op}}$$

is the covariant set-valued functor on the category $\underline{C}^{op}$ of algebras which assigns to each algebra $C^{op} \in \underline{C}^{op}$ its subset of elements of square 0. Since the full inclusion

$$\underline{C} \subseteq \mathcal{X} \subseteq \mathcal{S}^{\underline{C}^{op}}$$

preserves products and whatever exponentials may exist (for any small category $\underline{C}$), in order to verify our second axiom for $\mathcal{R}$, namely

$$R^D \cong R \times R$$

it suffices to know that $(\ )^D$, right adjoint to $(\ ) \times D$ the free or "tensor" product in Alg($\underline{A}$), exists in $\underline{C}$ and that the axiom holds there. In the case of algebraic geometry, where $\underline{A}$ consists only of polynomials with coefficients in some ground field k, this is indeed the case, in fact any algebra which is finite-dimensional as a k-vector space can be applied as an exponent in Alg $(\underline{A}_K)^{op}$, and $D = \mathrm{spec}(k[d])$, where $k[d] = k[t]/_{t^2}$ is two-dimensional. But it should be possible to take $\underline{A}$ as the algebraic theory of all real- analytic functions or of all C^∞ real functions of n variables. [In 1978 Eduardo Dubuc succeeded in constructing an $\mathcal{X}$ satisfying both of our two axioms and containing as a full subcategory

the category of all real C^{∞} manifolds; the extent to which it is generated by (the dual of) the category of all finitely - presented C^{∞}-algebras is still unclear to me at this writing}
There should be many algebraic theories $\underline{A}$ intermediate between only polynomials as operations and all C^{∞} functions as operations, perhaps satisfying some suitable closure conditions, in particular the $\underline{A}$ generated by cos, sin, exp, e^{-1/x^2} {Anders Kock has studied the closure condition that with each f $(\underline{x},t)$ in $\underline{A}$ there is also contained in $\underline{A}$ the unique continuous $f_1(\underline{x},t,h)$ such that $f(\underline{x},t+h) = f(\underline{x}) + f_1(\underline{x},t,h).h$ for all real $\underline{x}$, t,h. The inverse of this condition would also seem interesting }.

{Some feel that a geometrical category $\mathcal{X}$ should not require a category as big as $\underline{C}$ to generate it, nor should it satisfy the topos exactness condition that monic epics are isomorphisms, but rather should be generated (weakly) more nearly by points in a narrow sense. If

$$\mathcal{B} \subset \underline{C} \subset \mathcal{X}$$

are categories where $\mathcal{B}$ (weakly) generates $\underline{C}$, $\underline{C}$ generates $\mathcal{X}$, and $\mathcal{X}$ is complete cartesian closed, then the full subcategory $\mathcal{Y}_{\mathcal{B}}$ of $\mathcal{X}$ weakly generated by $\mathcal{B}$, can be defined to consist of all Y such that for any distinct $C \rightrightarrows Y$ with C in $\underline{C}$, there is $E \longrightarrow C$, with $E \in \mathcal{B}$ such that $E \longrightarrow C \rightrightarrows Y$ are different. $\mathcal{Y}_{\mathcal{B}}$ is closed under inverse limits and $\mathcal{X}$ -subobjects, so is epi-reflective in $\mathcal{X}$. In fact $\mathcal{Y}_{\mathcal{B}}$ is closed under exponentiation and contains $\underline{C}$ as a full subcategory, as well as being weakly generated by $\mathcal{B}$. The interest of this general construction for us is that an appropria Nullstellensatz for $\underline{A}$ would tell us that the dual $\mathcal{D}$ of the special category $\mathcal{W}$ of finite dimensional algebras defined earlier,

$$\mathcal{W}^{op} \xrightarrow[\approx]{D} \mathcal{D}$$

actually does weakly generate $\underline{C}$. Since $\underline{C} \subset \mathcal{Y}_{\mathcal{D}} \subset \mathcal{X}$ are full, $\mathcal{Y}_{\mathcal{D}}$ is more nearly a geometrical category than other cartesian closed categories generated (strongly) by $\mathcal{D}$ such as $\mathcal{S}^{\mathcal{W}}$. As pointed out in my 1972 Aarhus lectures, such categories as the latter retain only the <u>formal</u> aspects of the groups, spaces, etc from $\mathcal{X}$]

For any category $\underline{C}$ having finite products and split idempotents, an object $C \in \mathcal{S}^{\underline{C}^{op}}$ is representable iff the functor $(\ )^C$ has a right ajoint

$$\mathcal{S}^{\underline{C}^{op}} \xrightarrow[(\ )_C]{} \mathcal{S}^{\underline{C}^{op}}$$

In fact for any Y in $\mathcal{S}^{\underline{C}^{op}}$ we have for any S in $\underline{C}$

$$Y_C(S) = \mathrm{Nat}(S^C, Y)$$

and in particular if $S^C \in \underline{C}$ for all $S \in \underline{C}$ then

$$Y_C(S) = Y(S^C).$$

A subtopos $\mathcal{X}$ of $S^{\underline{C}^{op}}$ will be closed under $(\ )_C$ <u>provided</u> $(\ )^C$ preserves coverings. [Several people have recently pointed out that the foregoing is true (in the models) for $C = D(W)$, $W \in \mathcal{W}$. Thus for example we have the rule

$$\frac{X^D \longrightarrow Y}{X \longrightarrow Y_D}$$

In particular if $Y = R$ we find that there is a subobject L of R_D defined by the condition that the two induced actions of the multiplicative monoid of R agree. Then the gradient of a map $X \longrightarrow R$ can be interpreted not only as an element of

$$\mathrm{Hom}^{\bullet}_{R}(X^{D}, R) \subset R^{(X^{D})}$$

but also as a map

$$X \longrightarrow L \subset R_{D}.$$

In fact, there is a canonical map $R \xrightarrow{d} R_{D}$ which factors through L, and the gradient of any $X \xrightarrow{f} R$ can be computed as the composite

$$\mathrm{grad}(f) = df$$

Something like this feature exists also in the cartesian-closed category constructed by K.T. Chen in BAMS vol. 83, September 1977, even though the objects D and L do not exist since his category is weakly generated by 1 .]