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Model Theory and Topoi

A Collection of Lectures by Various Authors

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Introduction to Part I

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Part I of this volume consists of three of the first papers on functorial model theory, developing concretely the approach to algebraic logic according to which a "theory" (understood in a sense invariant with respect to various "presentations" by means of particular atomic formulas and particular axioms) is actually a category T having certain properties P and a model of T is any set-valued P - preserving functor. As a rough general principle, one could choose for P any collection of categorical properties which the category of sets satisfies, the choice then determining the "doctrine" of theories of kind P, which is thus a (non-full) subcategory of the category of small categories. For example, the doctrine of universal algebra thus springs from the fact that the category of sets has the property P of having finite cartesian products, while the doctrine of higher-order logic springs from the property of being a topos. The much-researched intermediate doctrine of (classical) first-order logic corresponds to the fact P that the category of sets has finite limits, complements of subsets, and images of mappings (related by the condition of being a "regular" category, which is essentially the logical rule $\exists x[A \wedge B(x)] \equiv A \wedge \exists x\, B(x)$ for A independent of x). The usual syntactical preoccupations of logic appear in the following way: once the logical operations and rules of inference are fixed (by the choice of P) the question arises of investigating free objects and hence presentation of arbitrary objects in the category of all P-categories T. But the often encountered suggestion that "syntax comes first" is refuted: the essential role of theories is to describe their models, and the same applies also to presentations of theories when the latter are needed for calculation. We often encounter and deal with groups for which we do not know or do not use any presentation: the same is true of theories.

Of course, for an arbitrary given P there is no guarantee of "completeness" in the usual sense, i.e. an arbitrary P-category T may fail to have enough models in the originally-envisioned category $\mathcal{S}$ of sets, sometimes paradoxically due to the fact that abstract sets are too "constant"; on the other hand it has become clear in the past decade that we are for reasons of geometry and analysis in fact interested in models in more general categories of variable sets such as sheaves over a topological space, Boolean-valued sets, algebraic spaces, permutation representations of a group, etc. - it is because of that that the interaction between the geometrical and logical aspects of general topoi has become an object of investigation, for example in the Bangor and Berlin parts of this volume.

Since a variable set may be partly empty and partly non-empty, the traditional model-theoretic banishment of empty models cannot be maintained, bringing to light a certain difficulty which the banishment obscured. Some claim that this difficulty is the "fact" that "entailment is not transitive", contrary to mathematical experience. However, the actual "difficulty" is that the traditional logical way of dealing with variables is inappropriate and hence should be abandoned. This traditional method (which by the way is probably one of the reasons why most mathematicians feel that a logical presentation of a theory is an absurd machine strangely unrelated to the theory or its subject matter) consists of declaring that there is one set I of variables on which all finitary relations depend, albeit vacuously on most of them; e.g. a binary relation on X is interpreted as $X^I \to 2$ depending vacuously on all but two of the variables in I. This is of course not totally absurd, since in the case of non-empty single-sorted structures, such an interpretation can be associated (in an infinite number of different but equivalent ways) to a correct interpretation. However, the fact that 2^{X^I} is a single Boolean algebra (claimed sometimes to be a "convenience") implies that propositional operators such as $\wedge, \vee, \Rightarrow$, applied indiscriminately to finitary relations, can be given a "meaning", a highly dubious "gain in generality", especially when, as noted above, the useful generalization to many sorts and/or partly empty domains is made.

Actually the (binary) propositional operators can only meaningfully be applied to (pairs of) relations having the <u>same</u> free variables. This may seem to prohibit such combinations as

$$(*) \qquad A(x,y) \wedge A(y,z) \Rightarrow A(x,z)$$

but consider the actual meaning: A denotes some subobject of the square X^2 of some sort X, and $(*)$ denotes a certain subobject of the cube X^3. The three projection maps $X^3 \overset{\to}{\underset{\to}{\to}} X^2$ induce three different substitution operators which to a binary relation A associate three different ternary relations $\sigma_{12}A$, $\sigma_{23}A$, $\sigma_{13}A$. Since conjunction and implication can meaningfully be applied to ternary relations, there is a ternary relation

$(\sigma_{12}A) \wedge (\sigma_{23}A) \Rightarrow \sigma_{13}A$ of which (*) is an abbreviation. Thus a syntax for presenting theories can be given in which propositional operators operate only among formulas with each fixed finite set of free variables, while substitution operators on an equal footing with quantifiers operate to change the set of free variables of a formula. These substitution operators have the structure (not of a monoid but) of a category with finite cartesian products; they need not consist only of tuples of projections, diagonal maps, etc. for if the presentation contemplates also function symbols, any m-tuple of terms in n free variables denotes a map $X^n \xrightarrow{f} X^m$ and hence induces a substitution f^* from m-ary relations to n-ary relations. If several basic sorts are considered, it is reasonable to consider that X^n, X^m are themselves further sorts V, W and that the m-tuple f of terms just referred to is simply another kind of term $V \xrightarrow{f} W$; it is then sensible to regard quantifications $\exists_f, \forall_f$ along an arbitrary such f , not only quantifications $\exists x, \forall x$ along projection maps $W \times X \to W$. The meaning of $\exists_f$, applied to a relation A of sort (or type) V is simply the relation $\exists_f A$ of type W which is the <u>image</u> of the composite map $A \rightarrowtail V \xrightarrow{f} W$; for any relation B of type W ,

$$\exists_f A \vdash_W B \quad \text{iff} \quad A \vdash_V f^* B$$

$$B \vdash_W \forall_f A \quad \text{iff} \quad f^* B \vdash_V A$$

are the rules of inference which characterize the two quantifiers as being respectively left and right adjoint to substitution. The subscripts V, W indicate that also entailments are only meaningful if both hypothesis and conclusion have the same set of free variables; the semantical meaning of entailment is inclusion between subjects of V (respectively of W).

It may be objected that in the above description of doctrines of theories the primacy of syntax has not been overturned since the determining property P must presumably be written in some language of categories. Since a general investigation of something like a "category of doctrines" has so far not seemed useful, the possible productive consequences of this contradiction, if any, are not known. However, one striking fact should be pointed out: While classes of theories with complicated definitions have been investigated in particular, the distinctive general classes which have actually been of interest, namely universal algebra, positive first-order logic, first-order logic, weak second order logic (= the "arithmetic universes" of Joyal), higher-order logic, etc, are all definable within an equational metatheory. More precisely the definition of such a doctrine amounts itself to a cartesian category (= category with finite limits) obtained by adjoining to the universal Horn theory of categories certain additional operators (usually denoting functors or natural transformations) whose domain is defined by equations, and imposing certain equations (which may hold only on equationally defined subvarieties) - usually in fact these equations express adjointness or distributivity of limits. Thus no disjunctions or existential quantifiers, nor any genuine occurence of universal quantifiers or implication, are involved in the definition of these doctrines. Here by a genuine occurence of a universal quantifier I mean something like the definition of a generator G

$$\forall x[G \xrightarrow{x} X \Rightarrow fx = gx] \vdash f = g$$

but <u>not</u> a universal Horn sentence

$$\forall x[A(x) \Rightarrow B(x)]$$

which can be replaced by a (free variable) inclusion of subobjects of X

$$A \vdash_X B$$

Even the "strong" conditions which distinguish a topos of "constant" sets from a general topos of variable sets,

(Axiom of Choice) For $X \xrightarrow{f} Y$,

if $1_Y \vdash \exists_f(1_X)$ then <u>there exists</u> $Y \xrightarrow{x} X$ with $f \circ x = 1_Y$

(Two-valuedness) For $1 \underset{\psi}{\overset{\varphi}{\rightrightarrows}} 1 + 1$

if $1 \vdash \varphi \vee \psi$ then $1 \vdash \varphi$ <u>or</u> $1 \vdash \psi$

do not involve genuine occurrences of universal quantification or implication, but do involve <u>there exists</u> and <u>or</u> on the right-hand side of an inference; hence, while not expressible in a cartesian (= Horn) metatheory, they are expressible in a <u>pretopos</u> metatheory so that the full algebraico-geometric method of coherent classifying topoi is applicable to them.

The paper by Orville Kean (his 1971 U. of Penn. dissertation) considers the case of theories which can be presented by axioms having the form of universal Horn sentences, i.e. the extension of "equational" universal algebra to the case in which some of the postulated identities between operations may hold only on "algebraic varieties" defined by equations between some other operations. Were one to consider an arbitrary set of "sorts", varying from theory to theory, rather than limiting oneself to the "one base set" for an algebra as is customary in universal algebra, and were one to allow further the possibility of partial operations whose domains of definition were such "algebraic varieties", then the appropriate condition on a category T would simply be: T is any small category with finite inverse limits (i.e. terminal object and pullbacks, hence finite products and equalizers, exist in T). Kean however takes care to analyze the further conditions on T corresponding to the restriction to one base sort on which all operations are defined. With or without these further conditions, the correct notion of model is simply any functor $T \to \mathbb{S}$ which preserves finite limits (i.e. which is "left exact") and the category of models is the category $Lex(T,\mathbb{S})$ of all such functors and all natural transformations between them. These categories of models retain the features from the equational universal algebra of being complete and having a set of generators which are "finitely-presented" objects in a categorically invariant sense, but in general fail to satisfy the two further properties characteristic of equational universal algebra that these generators can be taken as projective objects and that equivalence relations are effective (= "precongruences are congruences" in the terminology of my 1963 articles). The precise definition of "finitely-presented objects" can be found in Gabriel & Ulmer's Springer Lecture Notes volume 221, which also (implicitly) shows that "the functor Semantics has a functor Structure adjoint to it", but does not take any account of the relation with the logical concept of universal Horn axioms as Kean does. Another important feature of equational universal algebra which remains valid is the existence of left adjoints to the "algebraic" (syntactically induced) functors; i.e. if $T' \to T$ is any functor preserving finite limits between small categories having them, then the induced "forgetful" functor $\text{Lex}(T,\mathbb{S}) \to \text{Lex}(T',\mathbb{S})$ has a left adjoint. Here, since preferred "sorts" have less invariant significance in this doctrine, there is less motivation for requiring $T' \to T$ to preserve them even if they are there; this has of course the effect that such "forgetful" functors need not be faithful, but the added generality is mathematically very natural. For example, the functor $SO(2)$ from the category of commutative rings to the category of abelian groups is induced by a functor $T' \to T$ which does not preserve the base sort, since the base sort of the Horn theory of abelian groups is mapped to the subobject $\{\langle x,y\rangle | x^2 + y^2 = 1\}$ of the square of the base sort of the theory of commutative rings, but it is clear that this latter functor should be considered as an interpretation of the theory of abelian groups into the theory of commutative rings, indeed an interpretation "definable" within the doctrine of Horn theories.

The completeness of the category of models and the existence of left adjoints for induced functors are properties which in general do not carry over to theories more complicated than Horn theories, though it now seems that the adjoints may be recovered by allowing the "set-theory" $\mathbb{S}$ to vary along with the models (see remarks below).

The first detailed development of a purely categorical concept corresponding to full first-order theories was in the 1971 Dalhousie dissertation of Volger, on which the second article in this volume is based. The various sets of conditions on a category T which are considered in this article are corrections and improvements of a set conjectured earlier by me which exploited special properties of the Boolean case and coded formulas as morphisms into an object Ω which in various cases may be interpreted roughly as the object of sentences or the truth-value object. Volger considers throughout an arbitrary set of sorts, both because it is no more difficult and because various results, in particular his completeness theorem, then apply without change to type theory, which, whatever the exact notion of first-order theory T, means one which as a category is cartesian closed. Another feature which has remained invariant through the various experimentation which has gone on is the interpretation of quantifiers as functors adjoint to substitutions. Volger also outlines a modification of the completeness proof due to Andre Joyal which has played a role in the further unpublished development of the subject which has taken place since these papers were written.

These early calculations in categorical logic played a role in the development of the *elementary* theory of topoi (see, in addition to the present volume, SLN 274 and articles by Barr, Johnstone, W. Mitchell, Osius, and Paré in the Journal of Pure and Applied Algebra and the Bulletin of the AMS, Freyd's article in the Bulletin of the Australian Math Soc., for some of these developments) which in turn has affected the recent work in functorial model theory. In particular, using topoi, Kock and Mikkelsen (in the Victoria Symposium, SLN 369) generalized and clarified some basic constructions of non-standard analysis, which was one of the spurs to the further simplifications and application contained in Volger's second paper (1972) in this volume.

In the remainder of this introduction I sketch briefly some more recent developments in *geometric logic* wherein theories are modelled functorially in general topoi or in other words continuously variable models are studied. In this the doctrine of positive logic, i.e. $\exists,\wedge,\vee$, but no special attention to $\forall,\Rightarrow$, necessarily plays a distinguished role, since it is just this logic which is preserved under arbitrary continuous change of parameter space (the $\vee$ may be allowed to be infinitary) and also because an arbitrary Grothendieck topos can be viewed as the "classifying topos" for such a theory. However, full first-order logic can also be handled using the method due to Kripke and refined by Joyal and Freyd. More details can be found in my forthcoming paper in the Proceedings of the 1973 Bristol Logic meeting and in papers of Freyd, Johnstone, Joyal, Reyes and Wraith and by Benabou and his students.

In fact, important in algebraic geometry, that a sheaf of local rings is just a "local ring object" in the category of set-valued sheaves, remains valid when the theory of local rings is replaced by any many sorted theory in which only the logical operations $\wedge \vee \exists$ are considered and when sheaves are taken to mean objects in any topos. Here the truth of an existential statement or disjunction in the intrinsic logic of the topos is found by the adjointness rules of inference to mean locally, existence or locally, disjunction. The discrepancy between true (globally) and globally true (which is due to the fact that epimorphisms need not have sections and which gives rise to cohomology) may be exemplified by the fact that sheaf theoretically complex exponentiation is an epimorphism and hence the statement that the logarithm exists is true globally, but the actual existence takes place on a covering only. Intuitionistically, the same sort of relation between local and global holds even for a cubic. This class of theories may be considered to include any <u>classical</u> theory, since the negations of formulas may be considered as further atomic formulas and the axioms of negation considered as particular axioms rather than general axioms.

But the doctrine is basically intuitionistic, as is the intrinsic logic of the topoi where models are to be valued. The geometrically invariant condition on T to be a theory according to this doctrine is precisely that it should be a pretopos in the sense of Grothendieck-Verdier Exposé VI in Springer Lecture Notes Volume 270. The finite-covering topology on T leads to a topos $\underline{T}$ which, as pointed out by Reyes, has the property that for any topos $\underline{X}$ the category of continuous maps $\underline{X} \to \underline{T}$ is equivalent to the category of models in the "set theory" $\underline{X}$ of the theory T . The topos $\underline{T}$ is *coherent* in the sense of SLN 270 and all such arise from such theories; one may consider $\underline{T}$ as $\mathbb{S}[U]$, the "set theory" obtained by freely adjoining to the category of sets an indeterminate model U of T . Even for the theory T of equality, this construction is instructive; $\underline{T}$ in that case is the functor category $\mathbb{S}^{S_o}$ (where S_o is the category of finite sets) which is a non-trivial topos whose category of points is equivalent to the category of sets, and we have that for any topos $\underline{X}$, the sheaves on $\underline{X}$ are just the continuous functions from $\underline{X}$ into the (generalized) space $\underline{T}$ of sets.

The theorem of Deligne that every coherent topos has enough (set-valued) points is seen from the above discussion to be equivalent with the fact that every many-sorted intuitionistic theory taking account only of $\wedge, \vee, \exists$ has enough set-valued models. Further, the Kripke completeness theorem (preserving also $\forall, \Rightarrow$ when they exist) has been elegantly proved by Joyal in the invariant setting. The Kripke-Joyal Theorem constructs a model $\mathbb{S}^{\mathbb{D}} \to \underline{T}$ in a functor category rather than in sets $\mathbb{S}$; while the model itself preserves $\forall, \Rightarrow$ the "models" in $\mathbb{S}$ derived by evaluating at a given "stage of knowledge" $D \in \mathbb{D}$ usually do not.

Varying the topos in which we take models is quite essential for certain universal problems. For example consider the interpretation $T \to \overline{T}$ of the theory of commutative rings into the theory of local rings and consider any given ring A. The problem of finding a local ring $\overline{A}$ universal among all those to which A maps has no solution if we consider only one topos, but on the other hand if we allow the set theory to spread out, there is such a universal local ring in the topos called spec (A); thus the universal problem involves finding the natural domain of variation for the quantities in A, which will usually not be only the single point which corresponds to the topos of constant sets. When the topos of departure does not satisfy the axiom of choice, spec (A) does <u>not</u> have enough internal points (contrary to the incorrect statement in my paper for the 1970 International Congress) but Joyal has given a very simple <u>internal</u> construction of it using the notion of distributive lattice object. Since spec (A) is coherent* if $A \in \mathcal{S}$, Deligne's theorem yields enough external points for it when $\mathcal{S}$ does satisfy the axiom of choice. When the base topos of departure does not satisfy the axiom of choice, i.e. when it consists of variable sets varying in an organic fashion, a suitable formulation along these lines of a general completeness theorem for first-order theories in it has still to be found; such a formulation would presumably partly reflect the fact that in the real world consistency of a theory is not sufficient for the existence of models.

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*To prevent a possible delay in understanding the important exposé VI (SLN 270) of Grothendieck-Verdier cited above, it should be pointed out that their statement to the effect that separated coherent spaces are finite is incorrect; in fact these spaces are just the Stone spaces of arbitrary Boolean algebras, while arbitrary coherent topoi which are generated by their open sets are just "Stone spaces" of arbitrary distributive lattices. This is also a good place to point out that my statement in Springer Lecture Notes 274 that universal quantification in a topos leads to a triple is also incorrect; what was intended there is simply that universal quantification and infinite internal intersection satisfy the reasonable formal laws.