CONTINUOUSLY VARIABLE SETS; ALGEBRAIC GEOMETRY = GEOMETRIC LOGIC

F. William LAWVERE

University of Perugia, Perugia, Italy

The (elementary) theory of topoi, the fundamentals of which were outlined in Prof. Mac Lane's talk at this colloquium, (see also [6, 12, 13]) is a basis for the study of *continuously variable structures*, as classical set theory is a basis for the study of constant structures. The need for the autonomous development of such a theory may be doubted in view of the existence of representations of a variable structure, (e.g. a vector bundle or a family of curves) in terms of a domain of variation (considered as a constant structure such as a topological space) and a succession of constant structures, one for each 'point' in the domain of variation. But there is an analogy here with the notion of variable quantity, a notion which was taken quite seriously by the founders of analysis and which has not been 'eliminated' by set theory any more than continuity has been eliminated by the 'arithmetization of analysis' (which is just that and not analysis itself).

As Engels remarked in the period when set theory and the arithmetization of analysis did not yet dominate mathematical thinking, the introduction of the advance from constant quantities to variable quantities is a mathematical expression of the advance from metaphysics to dialectics, but many mathematicians continued to work in a metaphysical way with methods which had been obtained dialectically (*Anti-Dühring*, in the section on Quantity and Quality). The existence of a representation of a commutative ring of variable quantities in terms of functions on its spectrum does not eliminate the need for the theory of commutative rings (and indeed one of the ways of accounting for the differential structure of the variable quantities is precisely through the use of rings with nilpotent elements for which such representation in its classical form is not faithful). There are also useful concepts of variable quantity such as Schwartz distributions or Sato hyperfunctions, in which the 'domain of variation' is clearly ordinary space but just as clearly *not* the 'points' in it.

The characterization of motion as the presence of the same body in two places at the same time is only an *irresolvable* contradiction if we ignore that the metaphysical opposition between points and neighborhoods (introduced by the Platonic deification of points and revived by set theory) is not maintained in the practice even of mathematics. As Lenin affirmed in his Conspectus of Hegel's Lectures on the History of Philosophy (in the section on the Eleatic School) it is that characterization of motion which correctly expresses the continuity of time and space, whereas the concept of motion as the presence of a body one place at one time, in another place at a later time, describes only the result of motion and does not contain an explanation of motion itself. Every notion of constancy is relative, being derived perceptually or conceptually as a limiting case of variation[1] and the undisputed value of such notions in clarifying variation is always limited by that origin. This applies in particular to the notion of constant set, and explains[2] why so much of naive set theory carries over in some form into the theory of variable sets. Our inversion of the old theoretical program of modeling variation within eternal constancy has something in common with that of the intuitionists, though we consider variation generally, not only variation of mathematical knowledge; the internal logic of a topos is always concentrated in a Heyting algebra object. If this object happens to be Boolean, then the variation of the sets is (constant or) random in the sense that for every part b of the domain of variation the topos splits as a full product $\mathfrak{E} \cong \mathfrak{E}/_b \times \mathfrak{E}/_{b'}$, i.e., any motion over b and any motion over the complementary part can be combined into a total motion admitted by $\mathfrak{E}$, whereas for most topoi there is a continuity condition at the boundary of b; this is of course analogous to the contrast between continuous and measurable variable quantities.

There is a more profound connection than analogy between structure and quantity, as also was pointed out at this colloquium by Prof. Bernays. The primary subject matter of mathematics is the variation of quantity in time and space, but also this primacy has the nature of a first approxima-

[1] This remark is also relevant to non-standard analysis [4] which can also be clarified by topoi [11].

[2] 'Limited by that origin' has also a positive aspect.

tion, not only because occurring systems of quantities have structure, but also because of the fact that each material quantity is a quantity of something and hence has its own particular structure which we can hope to clarify mathematically.

Thus what I want to emphasize here about the theory of topoi is that it allows the passage from constant to variable sets (and back) and is a basis for studying relationships between (variable) quantities and (variable) structures. Since the theory arose from geometry and permits a deepening of analysis, it is striking that the axioms we arrived at are essentially (a categorical formulation of) the logician's definition of analysis: higher-order number theory. No general axiom of extensionality can be assumed but for a particular topos we may be able to discover a particular generalization of extensionality which is applicable (leading to a representation of the objects as sheaves) and if the topos is defined over a topos of constants in which the axiom of choice is valid, there *may* be enough points (leading to a representation of the objects as families of constant objects; this representation will however not account for the morphisms between objects without the further information of a left exact 'comonad', which generalizes the fact that the 'continuity' of a classical sheaf in its espace étale representation is not a property of the family of stalks but is the further information of a specified topology on their sum, and that morphisms between sheaves are represented only by those families of morphisms of constant objects which preserve these specified topologies). The close connection of the axiom of choice with the existence of points (primes) in algebraic geometry as well as with the existence of models in logic (below we will point out that models *are* points and show how both Krull's Theorem and the Gödel–Henkin–Kripke completeness theorem follow from Deligne's theorem on coherent topoi) is especially striking when we notice (Diaconescu) that the axiom of choice (in the form that all epimorphisms split) implies the law of the excluded middle and hence implies the constancy-randomness of sets as pointed out above; the falsity of the Sierpinski–Banach–Tarski paradox in the world is doubtless connected with the fact that material bodies are varying in a non-random fashion, and for similar reasons it is idealism to claim that something exists in the real world because its theory is consistent though of course the claim might be defended for a world of eternal thought. In order to extend the realm of direct applicability of the theoretical experience of set theory, part of our programme is the development of mathematics over an *arbitrary* base topos, (in particular one without the axiom of choice); a simple and beautiful example of this (discussed later in this paper) is a construction due to Joyal of the spectrum of a commutative ring without any use of primes (correcting an error in my paper, written in Nice, in which I mistakenly thought that enough such internal points would exist if only an intuitionistic definition was taken). Part of what follows was developed in discussions with and in unpublished lectures by André Joyal, Gonzalo Reyes, Jean Giraud, and Gavin Wraith.

The fact that the axiom of choice implies the law of the excluded middle does not mean that intuitionistic analysis is inconsistent, although the following simple argument, starting from del Ferro's theorem, might seem to show at first that it is

$$\forall y\ \exists x\ [y = x(x^2 - 3)],$$
$$\exists f\ \forall y\ [y = f(y)(f(y)^2 - 3)],$$
$$f : \mathbf{R} \to \mathbf{R} \quad \text{is continuous}$$
(since by Brouwer all functions are).

However, there is no such continuous f, although there is a covering of $\mathbf{R}$ by two open intervals $(-\infty, 1)(-1, \infty)$ on *each* of which continuous functions $f_{-\infty}, f_{\infty}$ can be defined which satisfy the equation. There are at least two lessons to be drawn from this: The choice 'functions' in intuitionism are not functions, i.e., do not preserve equivalence of Cauchy sequences though they are functions at the level of Cauchy sequences; this suggests a weaker 'axiom of choice' which *is* valid for the topos of sheaves on a *zero-dimensional* space (such as N^N), namely that (although not every object is projective) every object is the epimorphic image of some projective object (the 'some' could even be replaced by a functor); however for sheaves on a *space*, the space would at least have to be connected, no matter which definition of real-numbers object is taken, if Brouwer's theorem that the real numbers object is not the union of two proper disjoint subobjects is to hold. Kripke's method of modelling intuitionistic logic in a *presheaf* category, in which existential quantification commutes with evaluation at the 'stages' will probably not work for *higher-order* logic; the second lesson is that the more general 'commutation relation' for existential quantification involves *passing to a covering.* The latter is typical for general *sheaf* categories, as we will now explain more precisely for a more general class of categories.

It is a theorem (Mikkelsen and Paré, unpublished) that any topos (i.e., any category having finite inverse limits, a 'function space' functor $(\cdot)^A$ right adjoint to each cartesian product functor $A \times (\cdot)$, and a 'truth-value' object Ω uniquely classifying arbitrary subobjects of an arbitrary object A by characteristic functions $A \to \Omega$) also satisfies the following conditions characteristic of a *pretopos*: Besides a terminal object 1 and pullbacks, there exist a coterminal object O and finite coproducts (denoted by $+$) and these are preserved by pullback in the sense that for any $X \xrightarrow{f} Y$, $f^*(O_Y) = O_X$ and if $A_1 + A_2 \to Y$, then $f^*(A_1 + A_2) \cong f^*(A_1) + f^*(A_2)$ as objects over X; every equivalence relation $E \rightrightarrows A$ on an object A may be obtained by pulling back some $A \to B$ against itself; every morphism $A \to Y$ may be factored uniquely into an epimorphism followed by a monomorphism $A \to I \rightarrowtail Y$ and (especially important) this factorization is preserved on pulling back along any $X \to Y$. The subobject $I \rightarrowtail Y$ is called the image of $A \to Y$, and if $A_1 \rightarrowtail Y$, $A_2 \rightarrowtail Y$ are two subobjects, then the image of $A_1 + A_2 \to Y$ may be denoted by $A_1 \cup A_2 \rightarrowtail Y$. It follows that every epimorphism $A \to B$ is the coequalizer of the equivalence relation it induces and that every equivalence relation (though unlike for a topos, not necessarily *every* pair $A' \rightrightarrows A$ in a pretopos) has a coequalizer; there is clearly the derived rule $f^*(A_1 \cup A_2) = f^*A_1 \cup f^*A_2$. A typical example of a pretopos may be constructed as follows: Take any many-sorted first order theory involving at least the connectives $=, \wedge, \vee, \exists$ (logical equivalences and entailments may be used as axioms but not necessarily in formulas); let the objects of the category be arbitrary formulas of the theory and let the morphisms be (provable equivalence classes of) relations $A \xrightarrow{F} B$ which are provable functions, i.e.,

$$F(a,b) \vdash A(a), \qquad F(a,b) \vdash B(b),$$
$$\exists a\,[F(a,b) \wedge F(a,b')] \vdash b = b', \qquad A(a) \vdash \exists b\, F(a,b),$$

where a, b, b' are appropriate vector variables; adjoin coproducts and quotients formally if necessary. Then $A \xrightarrow{F} B$ is an epimorphism iff $B(b) \vdash \exists a\,[F(a,b)]$ and in general all existential quantifiers and images can be expressed in terms of each other. The fact that monic-epics in a pretopos are isomorphisms means that *unique* existence implies actual existence, but in general the axiom of choice fails because there are not enough constants. This last remark is connected with the non-trivial 'commutation rule' for existential quantification which we will now make more precise with the aid of an auxiliary class of objects.

Let $\mathcal{U}$ be any class of objects in a pretopos $\mathfrak{E}$. $\mathcal{U}$ could be the class of all objects for the purpose of the following proposition, but for example a useful condition on a topos $\mathfrak{E}$ is that there exists a single object the class of whose subobjects forms a suitable class $\mathcal{U}$. The objects of $\mathcal{U}$ may be variously interpreted as stages of subjective or objective time, as open sets, as rings of definition, etc; any morphism $U \xrightarrow{x} X$ of $\mathfrak{E}$ (whose domain U belongs to $\mathcal{U}$) may be called an 'element of X' defined at (or over) U, and if $U' \xrightarrow{t} U$ is any morphism $U' \xrightarrow{tx} X$ may be interpreted as x restricted to U' or the fate of the element x under the transition t. The suitability of $\mathcal{U}$ is expressed by the condition that $\mathcal{U}$ generates $\mathfrak{E}$ in the sense that any monomorphism $A \rightarrowtail^{f} Y$ in $\mathfrak{E}$ is an isomorphism provided it is '$\mathcal{U}$-surjective', where a morphism $A \xrightarrow{f} Y$ is called $\mathcal{U}$-surjective iff for any $U \xrightarrow{y} Y$ with U in $\mathcal{U}$ there exists $U \xrightarrow{a} A$ with $af = y$. For a pretopos this suitability is equivalent to another way of expressing $\mathcal{U}$-extensionality: If

$$A \underset{g}{\overset{f}{\rightrightarrows}} Y$$

in $\mathfrak{E}$ are such that $af = ag$ for all $U \xrightarrow{a} A$ and all U in $\mathcal{U}$, then $f = g$. In order to state the proposition we need one more definition: a class $\mathscr{C}$ of morphisms all having codomain A (but possibly various codomains) is said to *cover* A iff for any subobject $A' \rightarrowtail^{i} A$, we have an isomorphism

$$A' \underset{\approx}{\xrightarrow{i}} A$$

provided for every a in $\mathscr{C}$ we have $a \in A'$ (in the sense that there exists a' in $\mathfrak{E}$ with $a'i = a$). Clearly, if $\mathscr{C}$ is a subclass of $\mathscr{C}'$ and covers A, then the morphisms in $\mathscr{C}'$ with codomain A also cover A.

PROPOSITION 1. *Let $\mathfrak{E}$ be a pretopos and let $\mathcal{U}$ be a class of objects with respect to which $\mathfrak{E}$ satisfies $\mathcal{U}$-extensionality and let $X \xrightarrow{f} Y$ be a morphism of $\mathfrak{E}$. Let $V \xrightarrow{y} Y$ be an element of Y defined over V in $\mathcal{U}$. Then the validity of the formal condition $\exists x\, [X(x) \wedge xf = y]$ may not imply the existence of $V \xrightarrow{x} X$ with $xf = y$ (even if $V = 1$) but is equivalent with the validity of the categorical condition that the class of all those $U \xrightarrow{a} V$ for which there exists a commutative square*

$$\begin{array}{ccc} U & \xrightarrow{a} & V \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X & \xrightarrow[f]{} & Y' \end{array}$$

and for which U is in $\mathcal{U}$*, covers V. The morphism f is an epimorphism iff* $Y(y) \vdash \exists x\, [X(x) \wedge xf = y]$ *holds formally iff for every y with domain in* $\mathcal{U}$*, the class of a as above covers V, i.e., iff for every such y there is some cover* $\mathcal{C}$ *of V for each element* $\langle U, a\rangle$ *of which U is in* $\mathcal{U}$ *and the above square can be completed with an x. (If we simply took the pullback square, then we would have one epic a, but U would usually not be in* $\mathcal{U}$*.)*

As another example of the Proposition 1, consider the fact that the complex logarithm exists and yet does not exist. Logically speaking this contradiction was solved by passing to a deeper stage of knowledge, geometrically speaking by passing to a covering (the first integer cohomology group shows that the contradiction was not vacuous). Here we take $\mathfrak{E}$ as the category of set-valued sheaves on a topological space (such as an open set in the plane) and the class $\mathcal{U}$ of open subsets of the space is suitable; let X be the sheaf of complex-valued continuous functions and Y the sheaf of non-vanishing complex-valued continuous functions and take for f the exponential mapping. Then f is an epimorphism in $\mathfrak{E}$ (i.e., $\forall y\ \exists x$ is true with *formal* variables x, y) however (taking, say, $V = 1$, i.e., the whole space) for given y (i.e., a non-vanishing function defined on the whole space), there is no x with $xf = y$. But there is an open covering U_i of 1 with x_i such that $x_i f = y|U_i$.

A pretopos need not admit an internal universal-quantification operator, but if it does (as for example any topos does) then the 'commutation relation' for $\forall$ relative to a suitable $\mathcal{U}$ is the same one familiar from Kripke models and from forcing, i.e., the truth of a universal statement at V involves all elements defined over *all* U with $U \to V$ and U in $\mathcal{U}$, not only all elements defined over V.

The close relationship between the logical and geometrical ways of solving existential contradictions may be further illuminated by another simple proposition in which we change the global domain of variation (i.e., adjoin a variable 'constant'). The opposition between global, eternal elements $1 \to X$ and elements $U \to X$ with an arbitrary domain of definition is not metaphysically fixed. Consider the category $\mathfrak{E}/U$ whose objects are 'over U', i.e., objects of $\mathfrak{E}$ equipped with a structural morphism to U and whose morphisms are commutative triangles (i.e., morphisms of $\mathfrak{E}$ which respect the structural morphisms). Then the terminal object 1_U of $\mathfrak{E}/U$ is just the identity morphism of U and if we consider $X_U = U \times X$ as an object of $\mathfrak{E}/U$ (namely as the 'constant' X vacuously varying over U; this may amount to a restriction or an expansion of the original domain of variation according to the particular U), then the eternal elements $1_U \to X_U$ of X in the sense of $\mathfrak{E}/U$ are just the $U \to X$ defined over (or at) U in the sense of $\mathfrak{E}$.

PROPOSITION 2. *For any object U of $\mathfrak{E}$, $\mathfrak{E}/U$ is a pretopos if $\mathfrak{E}$ is. The functor*

$$\mathfrak{E} \xrightarrow{U\times(\cdot)} \mathfrak{E}/U$$

preserves the pretopos structure and the internal universal-quantification operator if $\mathfrak{E}$ has it (as well as the topos structure if $\mathfrak{E}$ has it). The object U has, after passing into $\mathfrak{E}/U$ by the functor, always a canonical eternal element (in terms of $\mathfrak{E}$, this global element is just the diagonal map $U \to U \times U$). If $U \to 1$ is an epimorphism in $\mathfrak{E}$, then the functor is faithful.

Note that U may have had no global elements in $\mathfrak{E}$. One geometrical example of the above proposition involves extending the ring of definition; then U is just the spectrum of the extended ring. If we recall the correspondence between pretopoi and many-sorted intuitionistic first-order theories, we see that the above proposition implies the well-known lemma on consistently adjoining constants; namely $U \to 1$ epic just means that $\vdash \exists u\, [U(u)]$ whereas a morphism $1 \to U$ is really a constant satisfying U; the construction of $\mathfrak{E}/U$ amounts to adjoining the 'constant' d and the axiom $\vdash U(d)$ and closing to obtain a theory of the same kind as before (i.e., to maintain the pretopos nature).

Recall that a continuous map (geometric morphism) $\mathfrak{X} \xrightarrow{f} \mathfrak{Y}$ between topoi is just a functor having a left exact left adjoint f^*. Given a fixed topos $\mathfrak{S}$, by a topos defined over $\mathfrak{S}$ is meant a topos $\mathfrak{X}$ together with a given continuous map $\mathfrak{X} \to \mathfrak{S}$, and by a continuous map over $\mathfrak{S}$ is meant a continuous map $\mathfrak{X} \to \mathfrak{Y}$ which commutes up to a given equivalence of functors with the given maps $\mathfrak{X} \to \mathfrak{S}$, $\mathfrak{Y} \to \mathfrak{S}$. If

$$\mathfrak{X} \underset{g}{\overset{f}{\rightrightarrows}} \mathfrak{Y}$$

are two continuous maps over $\mathfrak{S}$, then by a morphism $f \to g$ is meant any natural transformation $f^* \to g^*$ which reduces to the identity on $\mathfrak{S}$; thus there is a *category* $\mathrm{Top}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$ of continuous maps from $\mathfrak{X}$ to $\mathfrak{Y}$; seemingly all that can be affirmed about it in general is that it has colimits over *filtered* category objects from $\mathfrak{S}$ (e.g. over directed poset objects in $\mathfrak{S}$). In particular, $\mathrm{Top}_{\mathfrak{S}}(1_{\mathfrak{S}}, \mathfrak{Y})$, the category of sections of the given structural map of $\mathfrak{Y}$, is called the category of ($\mathfrak{S}$-valued) points of $\mathfrak{Y}$. If $\mathfrak{Y}$ is the category of internal $\mathfrak{S}$-valued sheaves on a complete Heyting-algebra object in $\mathfrak{S}$, the morphisms in the category of points of $\mathfrak{Y}$ generalize the usual partial ordering of points of a not-necessarily T_1-space. The category of points may be empty, even if $\mathfrak{S}$ is 'the' category of constant abstract sets–for example, if $\mathfrak{Y}$ is the category of sheaves on a complete Boolean-algebra object (in $\mathfrak{S}$) which is *atomless*; of course we can then generalize the 'point' analysis of $\mathfrak{Y}$ by considering 'points' (i.e., continuous maps) of $\mathfrak{Y}$ defined over a suitable class of $\mathfrak{X}$'s and below we describe a precise theorem of Barr to that effect.

The sense in which a topos $\mathfrak{X}$ equipped with a continuous map $\mathfrak{X} \xrightarrow{p} \mathfrak{S}$ is 'defined over' the topos $\mathfrak{S}$ has two aspects. The 'closed category' aspect (exploited in [18]) is that we can define

$$\mathfrak{X}(A, X) = p_*(X^A)$$

for any two objects A, X of $\mathfrak{X}$, so that the hom sets of $\mathfrak{X}$ are not merely abstract sets but are enriched to be objects of $\mathfrak{S}$, and in particular we may consider p_* itself as the functor represented by $A = 1$. The other aspect is that for any object S of $\mathfrak{S}$ we can define

$$\mathfrak{X}^S = \mathfrak{X}/p^*(S)$$

and consider the latter as the category of S-indexed families of objects of $\mathfrak{X}$; such a family (i.e., an object of $\mathfrak{X}^S$) should be thought of as '$\mathfrak{S}$-smoothly' indexed. Usual formal operations on families remain within these, for example substituting along any 'change of index set' $S' \to S$ in $\mathfrak{S}$, internal coproduct and product

$$\mathfrak{X}^S \underset{\prod_S}{\overset{\Sigma_S}{\rightrightarrows}} \mathfrak{X},$$

etc. One may consider the structure on $\mathfrak{X}$ consisting of the notion of families as an *atlas* on $\mathfrak{X}$ with models in $\mathfrak{S}$, and these operations on families as coordinate transformations; this notion of atlas will also be realized for many categories $\mathfrak{X}$ over $\mathfrak{S}$ which are *not* topoi such as the category ab($\mathfrak{S}$) of abelian group objects in $\mathfrak{S}$, the category top($\mathfrak{S}$) of topological space objects in $\mathfrak{S}$, the category cat($\mathfrak{S}$) of ('small') category objects in $\mathfrak{S}$, etc., and in general we should consider between such 'large' categories 'over' $\mathfrak{S}$ only those functors which respect the atlas structures (which apparently any functor definable within the set theory $\mathfrak{S}$ will automatically do).

The general facts about topoi over a base topos $\mathfrak{S}$ depend of course on $\mathfrak{S}$, but here we will discuss mainly the three cases of any topos $\mathfrak{S}$, a topos $\mathfrak{S}$ having a natural-numbers object N, and any topos $\mathfrak{S}$ satisfying the axiom of choice (the conjunction of the last two conditions we may roughly identify with Boolean-valued models $\mathfrak{S}$ of Zermelo set theory weakened to *bounded* comprehension). Naturally, more can be affirmed if we assume that one or both of $\mathfrak{X}$, $\mathfrak{Y}$ satisfy some 'smallness' condition relative to $\mathfrak{S}$; we will consider mainly three such conditions.

The first smallness condition reflects a very important construction in the classical sheaf theory, but has not been investigated much in topos theory and in particular, I do not know an internal characterization of those maps $\mathfrak{Y} \to \mathfrak{S}$ which satisfy it. The external form is this: for every topos $\mathfrak{X}$ over $\mathfrak{S}$, there is an object $\Gamma_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$ in $\mathfrak{X}$ and an equivalence of categories

$$\mathrm{Top}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y}) \cong \mathfrak{X}(1, \Gamma_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y}))$$

and in particular

$$\mathrm{Top}_{\mathfrak{S}}(\mathfrak{X}/X, \mathfrak{Y}) \cong \mathfrak{X}(X, \Gamma_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y}))$$

for any object X of $\mathfrak{X}$. Then $\Gamma_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$ is determined as the 'sheaf of $\mathfrak{Y}$-valued continuous functions (over $\mathfrak{S}$) defined in $\mathfrak{X}$'. This will exist at least in the case that $\mathfrak{Y}$ is the topos of sheaves on a complete Heyting algebra in $\mathfrak{S}$ (for example on the open sets of a topological space object in $\mathfrak{S}$). The fact that $\Gamma_{\mathfrak{S}}(1_{\mathfrak{S}}, \mathfrak{Y})$ is an object of $\mathfrak{S}$ is the (quite restrictive) condition that $\mathfrak{Y}$ has only a 'set' of points over $\mathfrak{S}$; if $\mathfrak{S}$ is thought of as the sheaves on a base space more general than a point, it is more usual to call $\Gamma_{\mathfrak{S}}(1_{\mathfrak{S}}, \mathfrak{Y})$ the *sheaf of sections* of $\mathfrak{Y} \to \mathfrak{S}$ rather than calling it the 'set of points'. Also neglected has been the study of the intrinsic structure of the objects $\Gamma_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$; for example they have a category structure (it need not reduce to a poset even though it is small; see [1, Vol. I, pp. 479–490 on 'étendues']) and a topology (richer even in the classical case than the espace étalé topology, which in the intrinsic sense of $\mathfrak{X}$ is *discrete*) derived from the truth-value object of $\mathfrak{Y}$.

For example, if $\mathfrak{S}$ has a natural numbers object, then it also has an object $\mathbf{R}_{\text{Cauchy}}$ of Cauchy-real numbers, and $\mathfrak{Y} = \mathfrak{S}/\mathbf{R}_{\text{Cauchy}}$ is a topos over $\mathfrak{S}$. For any $\mathfrak{X}$ over $\mathfrak{S}$, $\Gamma_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$ is then the sheaf of *locally-constant* real valued functions defined in $\mathfrak{X}$; if $\mathfrak{X}$ happens to be 'locally connected over $\mathfrak{S}$' in the sense that the structural map $\mathfrak{X} \xrightarrow{p} \mathfrak{S}$ is *essential*, i.e., there is another functor $\mathfrak{X} \xrightarrow{\pi_0} \mathfrak{S}$ left adjoint to the left adjoint p^*, then the sheaf of locally constant real-valued functions in $\mathfrak{X}$ is just the object of Cauchy reals *calculated in the sense of* $\mathfrak{X}$. On the other hand, the object of Dedekind reals as calculated in $\mathfrak{X}$ will, at least in many cases, be $\Gamma_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{R})$, where $\mathfrak{R} = \text{sh}(\mathbf{R}, \mathfrak{S})$ is the 'usual' category of sheaves over $\mathbf{R} \in \mathfrak{S}$ with its usual topology; thus the Dedekind reals in $\mathfrak{X}$ typically correspond to continuous real functions on a topos $\mathfrak{X}$ while the Cauchy reals are only the locally constant ones (even if the two definitions happen to agree in the base topos $\mathfrak{S}$). Thus a particular quantity (varying over $\mathfrak{X}$) may be represented by a pair of adjoint functors (i.e., continuous map of topoi)

$$\mathfrak{X} \longrightarrow \mathfrak{R}$$

and other kinds of quantities can also be so represented by considering a different topos in place of $\mathfrak{R}$. Truth values may be considered a kind of quantity, and the category $\mathfrak{S}^2$ of morphisms in $\mathfrak{S}$ plays the role of the Sierpinski space relative to $\mathfrak{S}$ in the sense that continuous maps $\mathfrak{X} \rightarrow \mathfrak{S}^2$ correspond to morphisms

$$1 \rightarrow \Omega_{\mathfrak{X}} = \Gamma_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{S}^2) \quad \text{in } \mathfrak{X}.$$

If Π is an abelian group in $\mathfrak{S}$, we might agree to consider even an element of the cohomology group $H^1(\mathfrak{X}, \Pi)$ as a 'quantity varying over $\mathfrak{X}$'; at any rate such an element can be represented by a continuous map

$$\mathfrak{X} \longrightarrow \mathfrak{S}^{\Pi}$$

of topoi over $\mathfrak{S}$ (where $\mathfrak{S}^{\Pi}$ is the category of Π-sets in $\mathfrak{S}$), or again by the structure of a principal homogenous space in $\mathfrak{X}$. We will see below that quite general kinds of *structures* (e.g., first order structures) varying over $\mathfrak{X}$ can *also* be represented by continuous maps

$$\mathfrak{X} \longrightarrow \mathfrak{T}$$

where $\mathfrak{T}$ depends on the particular kind of structure but $\mathcal{X}$ is any topos over $\mathfrak{S}$; but since typically there are a 'proper class' of structures of kind $\mathfrak{T}$, there will usually *not* exist objects $\Gamma_{\mathfrak{S}}(\mathcal{X}, \mathfrak{T})$ in $\mathcal{X}$. First we discuss another condition on topoi over $\mathfrak{S}$.

The second smallness condition is essentially that which is usual in the Grothendieck–Giraud–Verdier theory (relativized to $\mathfrak{S}$) and internalizes the notion of a *set* $\mathcal{U}$ of generators as discussed above; first we discuss a special case. Intuitively, $\mathfrak{Y}$ over $\mathfrak{S}$ is the category of sheaves for a complete Heyting-algebra object in $\mathfrak{S}$ iff 'relative to $\mathfrak{S}$' $\mathfrak{Y}$ is generated by the subobjects of 1 (i.e., by the class $\mathcal{U}$ of 'open sets' of $\mathfrak{Y}$). A compact internal expression of this idea was found by William Mitchell, who proved the following precise proposition:

PROPOSITION 3. *If $\mathfrak{Y} \xrightarrow{p} \mathfrak{S}$ is a continuous map of topoi, then $\mathfrak{Y}$ is the category of internal $\mathfrak{S}$-valued sheaves on a complete Heyting-algebra object of $\mathfrak{S}$ (namely $p_*(\Omega_{\mathfrak{Y}})$ iff for every injective object Y of $\mathfrak{Y}$ the canonical adjunction morphism*

$$p^*(p_*(Y)) \longrightarrow Y$$

is an epimorphism. In fact it suffices to consider those (automatically injective) Y which are the partial-morphism classifiers $\tilde{Z}$ of arbitrary objects Z.

The general case of $\mathfrak{Y}$ generated, roughly speaking, by the class of all subobjects of some object G was conjectured in the following form by Mitchell and proved by Diaconescu; the original version (for $\mathfrak{S}$ = a Grothendieck universe) is a theorem of Giraud, who later proved a relative version for $\mathfrak{S}$ a Grothendieck topos [14].

THEOREM 4. *$\mathfrak{Y} \xrightarrow{p} \mathfrak{S}$ is the global sections functor for the category of sheaves over some site in $\mathfrak{S}$ iff there exists an object G in $\mathfrak{Y}$ such that for all objects Y of $\mathfrak{Y}$ the canonical morphism*

$$G \times p^*(p_*(\tilde{Y}^G)) \longrightarrow \tilde{Y}$$

obtained by composing evaluation with the adjunction is an epimorphism in $\mathfrak{Y}$ (where $\tilde{Y}$ denotes the object such that $\mathfrak{Y}(X, \tilde{Y})$ is the set of partial morphisms from any X to Y).

For the purpose of this paper let us say that $\mathfrak{Y}$ is *bounded* over $\mathfrak{S}$ if this second smallness condition (i.e., the two equivalent conditions on p in the above theorem) holds. The following theorem of Hakim–Giraud (in the case of Grothendieck topoi) was proved for arbitrary $\mathfrak{S}$ by Radu Diaconescu in his thesis [5].

THEOREM 5. *If $\mathfrak{Y}$ is bounded over $\mathfrak{S}$, then for any topos $\mathcal{X}$ over $\mathfrak{S}$, the pullback $\mathcal{X} \times_{\mathfrak{S}} \mathfrak{Y}$ (= cartesian product in the up-to-isomorphism sense of the 2-dimensional category* $\mathrm{Top}_{\mathfrak{S}}$) *exists as a topos defined over $\mathfrak{S}$.*

Another property of $\mathfrak{Y}$ bounded over $\mathfrak{S}$ proved in [1, Vol. I, pp. 400–401]: The category of points contains a family indexed by an object of $\mathfrak{S}$ such that every point is a filtered $\varinjlim$ of points in the family; in view of the results of Reyes to be discussed below, this is a geometric version of a general Löwenheim–Skolem theorem.

The following illuminating fact about topoi (long known for the case $\mathfrak{S}$ = constant sets) was (conjectured by me and) proved by Gavin Wraith for any base topos having a natural-numbers object.

THEOREM 6. *Suppose $\mathfrak{S}$ is a topos having a natural numbers object. Then there is a topos $\mathfrak{S}[T]$ over $\mathfrak{S}$ 'obtained by adjoining an indeterminate set T' such that for any topos $\mathcal{X}$ over $\mathfrak{S}$ there is an equivalence*

$$\mathrm{Top}_{\mathfrak{S}}(\mathcal{X}, \mathfrak{S}[T]) \xrightarrow{\sim} \mathcal{X}$$

of categories (defined by $f \rightsquigarrow f^*(T)$). Specifically, $\mathfrak{S}[T]$ is the (internal) functor category $\mathfrak{S}^{S_0}$, where S_0 is a category object in $\mathfrak{S}$ which may be interpreted as the category of finite sets with $S_0 \xrightarrow{T} \mathfrak{S}$ interpreted as the full inclusion.

In particular, the category of points of $\mathfrak{S}[T]$ is equivalent to $\mathfrak{S}$, so that if we accept Grothendieck's dictum that a topos is a generalized space, $\mathfrak{S}[T]$ is the space of all sets, with a non-trivial topology! Indeed, if x is a point of $\mathcal{X}$ and if $\mathcal{X} \xrightarrow{f} \mathfrak{S}[T]$, then the composite $\mathfrak{S} \xrightarrow{x} \mathcal{X} \xrightarrow{f} \mathfrak{S}[T]$ is a point of $\mathfrak{S}[T]$, i.e., a set. The topology being non-trivial signifies that $x \rightsquigarrow f(x)$ is continuous iff f is a *sheaf* on the 'space' $\mathcal{X}$, i.e., corresponds to an object of $\mathcal{X}$. The last sentence is inaccurate in two respects: $\mathcal{X}$ need not have enough points, and more profoundly, $\mathfrak{S}[T]$ does not have enough 'open sets', so that to specify the 'topology' on the space of sets we have to consider all the 'open sets with multiplicity' = all the sheaves, which correspond to all the functors $S_0 \xrightarrow{E} \mathfrak{S}$; for the just stated reason, continuity of f is not a 'property' of the point-mapping but is an additional structure consisting of the specification of the (generalized open set, i.e., object in $\mathfrak{X}$) inverse image $f^*(E)$ of every such generalized open set E (this situation may be compared with current ideas in proof theory whereby properties are replaced by specified structures which 'prove' the properties). Actually, the theorem itself suggests that Grothendieck's dictum should be turned backward: a topos $\mathfrak{X}$ is the 'algebra of continuous (set-valued) functions' on a generalized space, not the generalized space itself. Naturally, the 'algebras' of variable sets are more profound than algebras of variable quantities, in the sense that their morphisms have two aspects f_*, f^* going in both the geometrical and algebraic directions; considering that the basic direction is that of f_* is essential in allowing the use of ideas from geometric experience in dealing with concepts whose relation to ordinary space is quite complicated, but f^* is usually more directly connected with the calculations which, rather than contemplation of pictures, are the main concern of mathematics, even of geometry. The foregoing sentence is a conceptual justification of the convention that the morphisms in $\mathrm{Top}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$ are natural transformations $f^* \to g^*$ rather than natural transformations $f_* \to g_*$; formal justification is given by the above theorem and its generalizations which follow, for with the opposite convention all these theorems would contain unnecessary dualizations. (Speculatively, it may become necessary to consider still more general 'generalized spaces' which are not determined by a single topos of variable sets, just as in algebraic geometry one considers non-affine spaces not determined by a single ring of variable quantities; this would lead to a large 2-dimensional 'topos' containing $\mathrm{Top}_{\mathfrak{S}}$ as a full subcategory with $\mathfrak{S}^{S_0}$ playing the role of the 'affine line'.)

The above is a case of the following theorem (one generalization of which, to arbitrary functor categories $\mathfrak{S}^{C^{op}}$, is given by Diaconescu [5]; however, the more interesting generalization for us here will be to consider 'larger' categories $\boldsymbol{C}$ with more structure and correspondingly 'smaller' topoi of *sheaves* on $\boldsymbol{C}$ rather than all presheaves).

Theorem 7. *Let $\boldsymbol{C}$ be a category object in a topos $\mathfrak{S}$ such that $\boldsymbol{C}$ has finite (inverse) limits. For any topos $\mathfrak{X}$ over $\mathfrak{S}$, let $\mathrm{Alg}_{\mathfrak{S}}(\boldsymbol{C}, \mathfrak{X})$ denote the category of $\mathfrak{S}$-smooth left-exact functors $\boldsymbol{C} \to \mathfrak{X}$. Then there is an equivalence of categories*

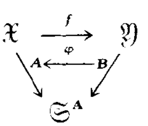
$$\mathrm{Top}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{S}^{C^{op}}) \cong \mathrm{Alg}_{\mathfrak{S}}(\boldsymbol{C}, \mathfrak{X}).$$

(Here, since $\mathfrak{X}$ is a topos equipped with a map $\mathfrak{X} \xrightarrow{p} \mathfrak{S}$, the problem of explaining '$\mathfrak{S}$-smooth' can be side-stepped by defining $\mathrm{Alg}_{\mathfrak{S}}(\boldsymbol{C}, \mathfrak{X})$ as being (the part consisting of those functors preserving finite limits) in the internal (to $\mathfrak{X}$!) pre-sheaf category $\mathfrak{X}^{p^*(C)}$. But for a general theory of large categories $\mathfrak{X}$ over a base topos $\mathfrak{S}$, we would have to consider as given categories $\mathfrak{X}^S$ of smoothly S-indexed families of objects of $\mathfrak{X}$ (which here can be defined as $\mathfrak{X}^S = \mathfrak{X}/p^*(S)$) and then we would see that a smooth functor $\boldsymbol{C} \to \mathfrak{X}$ really involves a family of functors $\mathfrak{S}(S, \boldsymbol{C}) \to \mathfrak{X}^S$ compatible with change of S, for at least an adequate family of objects S of $\mathfrak{S}$; $S = 1$ alone will not be adequate unless $\mathfrak{S}$ or $\boldsymbol{C}$ is very special.)

The interpretation of the above theorem is as follows: $\boldsymbol{C}$ is a many-sorted partial equational (finitary) theory (universal Horn theory), and $\mathrm{Alg}_{\mathfrak{S}}(\boldsymbol{C}, \mathfrak{X})$ is the category of models of $\boldsymbol{C}$ in $\mathfrak{X}$; in particular, $\mathrm{Alg}_{\mathfrak{S}}(\boldsymbol{C}, \mathfrak{S})$ is the category of models of $\boldsymbol{C}$ in $\mathfrak{S}$. The pre-sheaf topos $\mathfrak{S}^{\boldsymbol{C}^{\mathrm{op}}}$ may be considered as the set theory obtained by adjoining an indeterminate model of $\boldsymbol{C}$ to the set theory $\mathfrak{S}$, or as the (algebra of continuously variable 'sets' over the) generalized space whose points are all the ($\mathfrak{S}$-) models of $\boldsymbol{C}$. For example, if $\boldsymbol{V}$ is the category of finite-dimensional (say rational) vector spaces in $\mathfrak{S}$, then a topos morphism

$$\mathfrak{X} \longrightarrow \mathfrak{S}^{\boldsymbol{V}}$$

over $\mathfrak{S}$ may be considered either as a sheaf of vector spaces over the 'space' $\mathfrak{X}$, as a vector space object in the topos $\mathfrak{X}$, or as a map assigning vector spaces (in $\mathfrak{S}$) to points of $\mathfrak{X}$ in a *continuous* manner. Here we have used $\boldsymbol{V}^{\mathrm{op}} \cong \boldsymbol{V}$; starting from an algebraic standpoint it may be more natural to start with a category $\boldsymbol{A} = \boldsymbol{C}^{\mathrm{op}}$ having finite colimits. For example, if $\boldsymbol{A}$ is the category of all finitely presented commutative rings in $\mathfrak{S}$, then $\mathfrak{S}^{\boldsymbol{A}}$ is the 'space of all rings'[1]. $\mathrm{Top}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{S}^{\boldsymbol{A}}) \cong \mathrm{Ann}(\mathfrak{X})$, and the category Top $\mathrm{Ann}_{\mathfrak{S}}$ of all ringed topoi over $\mathfrak{S}$ may be considered as the '2-dimensional comma category' $\mathrm{Top}//\mathfrak{S}^{\boldsymbol{A}}$ whose morphisms are all the (non-commutative) triangles



[1] In calling $\mathfrak{S}^{\boldsymbol{A}}$ the 'space of all rings' we are thinking of its *points*; if we think of its *sheaves* (= objects), it might also be called the 'ring of all spaces'–i.e., a topos containing a *line* R and all cartesian products, disjoint sums, equationally defined subspaces, quotients, etc. of this 'space'.

of continuous maps equipped with a morphism φ between continuous maps. Other natural choices for $\boldsymbol{C}$ are the theories of categories, closed categories (which seem especially interesting here as a generalization of commutative ring), pretopoi, topoi, all in the sense of structured *objects* in topoi $\mathfrak{X}$ over a topos $\mathfrak{S}$ with natural-numbers object.

However, partial equational (universal Horn) theories are not general enough, even for algebraic geometry where *local rings* play a dominating role. If $\boldsymbol{A}$ is the category of finitely presented rings then in $\boldsymbol{C} = \boldsymbol{A}^{\mathrm{op}} \subseteq \mathfrak{S}^{\boldsymbol{A}}$ the indeterminate ring R corresponds to the ring $\mathscr{Z}[t]$ in $\boldsymbol{A}$ with one indeterminate element, and the subobject U of R representing the invertible elements corresponds to the ring $\mathscr{Z}[t, t^{-1}]$ in $\boldsymbol{A}$. The condition for a ring object R in a topos $\mathfrak{X}$ to be *local* is that for any pair f, g of its elements

$$[f \in U] \vee [g \in U] \equiv \exists a\ \exists b\ [a \cdot f + b \cdot g \in U],$$

i.e., an isomorphism of two subobjects of $R \times R$ in $\mathfrak{X}$. But since these two subobjects are not equationally defined, they do not exist as subobjects of R in $\boldsymbol{C}$; thus we enlarge $\boldsymbol{C}$ to a *pretopos* then adjoin the above condition to obtain another pretopos $\boldsymbol{E}$. (I do not know a simple direct algebraic description of $\boldsymbol{E}$.) Then we consider a topos $\mathfrak{Z}$ which may be defined as the largest subcategory of $\mathfrak{S}^{\boldsymbol{E}^{\mathrm{op}}}$ which contains the (Yoneda representation of) E and for which $E \to \mathfrak{Z}$ preserves finite coproducts and epimorphisms. $\mathfrak{Z}$ may also be described as the topos of *precanonical* sheaves on the pretopos $\boldsymbol{E}$ [1, Vol. II, §6]. Then there is an equivalence of categories

$$\mathrm{Top}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Z}) \cong \mathrm{Ann\ loc}(\mathfrak{X}),$$

where the right-hand side is the category of all local rings in $\mathfrak{X}$ and all ring homomorphisms between them, i.e., the category $\mathrm{Pretop}_{\mathfrak{S}}(E, \mathfrak{X})$ of $\mathfrak{S}$-smooth functors preserving the pretopos structure; I do not know if the local rings and *local* ring homomorphism in topos $\mathfrak{X}$ can be classified in a similar way by a single topos [9].

THEOREM 8 (Reyes). *For any many-sorted intuitionistic first-order theory* $\boldsymbol{E}$ *in which only the connectives* $=, \wedge, \vee, \exists$ *occur in the language although entailment may be used in the axioms, let* $\mathfrak{T}$ *be the topos of precanonical sheaves on the pretopos generated by the formulas. Then for any topos* $\mathfrak{X}$ *over sets* $\mathfrak{S}$,

$$\mathrm{Top}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{T}) \cong \mathrm{Mod}(E, \mathfrak{X}),$$

where morphisms of models are just natural transformations between functors preserving the pretopos structure.

The above is the basic theorem of a kind of 'geometric' logic which dualizes and extends algebraic logic in the sense that those continuous maps $\mathfrak{T}' \to \mathfrak{T}$ which correspond to relative interpretations $\boldsymbol{E} \to \boldsymbol{E}'$ of theories must at least preserve directed colimits.

If a pretopos $\boldsymbol{E}$ happens to have $\forall$ (and hence $\Rightarrow$) (e.g. any topos) in the sense that for each $X \xrightarrow{f} Y$ in $\boldsymbol{E}$ the operation of pulling back *subobjects* of Y to *subobjects* of X has a right adjoint $\forall_f$, then the 'models' in the above theorem are in general only *premodels* in the sense that they need not preserve $\forall$ (of course if E is classical in the sense that every subobject of every object has a strictly complementary subobject, then all premodels are models). I believe the following version of a construction of Kripke can be proved by methods of Joyal.

CONJECTURE 9. Let $\boldsymbol{E}$ be a pretopos with $\forall$ and let $\boldsymbol{P}$ be a sufficiently big set of premodels of E in $\mathfrak{X}$ considered as a *full* subcategory of Mod(E, $\mathfrak{X}$). Then the corresponding single premodel of E in $\mathfrak{X}^{\boldsymbol{P}}$ is actually a model. (This is possible because due to the action of the transition in $\boldsymbol{P}$, we have the Kripke commutation relation for $\forall$, i.e., $\forall$ in $\mathfrak{X}^{\boldsymbol{P}}$ is not preserved by evaluation at individual $p \in \boldsymbol{P}$.)

On the other hand, the existence of sufficiently big sets of premodels cannot in general be affirmed, unless $\mathfrak{X}$ satisfies the axiom of choice. To state Deligne's theorem we first state the characterization of topoi which arise as precanonical sheaves on some small pretopos. (This will be our third smallness condition). The term 'coherent' denotes a non-linear version of the concept in linear algebra which arose from the fact that in complex function theory the ring of holomorphic functions is not Noetherian but still satisfies a useful finiteness condition.

DEFINITION 10. A topos $\mathfrak{T}$ over $\mathfrak{S}$ is *coherent* iff its class of coherent objects is equivalent to a category object in $\mathfrak{S}$, is closed with respect to finite limits, and generates $\mathfrak{T}$ (over $\mathfrak{S}$). Here an object C is *coherent* iff it satisfies *both* the following conditions:

(1) for any *directed* poset I in $\mathfrak{S}$ and any system T_i in $\mathfrak{T}$,

$$\mathfrak{T}(C, \varinjlim_{i\in I} T_i) \longrightarrow \varinjlim_{i\in I} \mathfrak{T}(C, T_i)$$

in $\mathfrak{S}$,

(2) if $C_1 \to C$, $C_2 \to C$, where C_1, C_2 satisfy (1), then also $C_1 \times_C C_2$ satisfies (1).

THEOREM 11 (Deligne, see [1, Vol. II, §6]; a more categorical proof is due to Joyal). *If $\mathfrak{S}$ satisfies the axiom of choice and has a natural number object, then any topos $\mathfrak{T}$ coherent over $\mathfrak{S}$ has points. In fact there is K in $\mathfrak{S}$ and a continuous map $\mathfrak{S}^K \to \mathfrak{T}$ over $\mathfrak{S}$ whose inverse-image functor is faithful.*

In the same way that the above theorem is equivalent to the completeness theorem for first-order logic (in finitary languages) the following theorem is equivalent to a Boolean-valued completeness theorem for infinitary logic, since as Reyes and Barr[22] have affirmed, *every* Grothendieck topos $\mathfrak{Y}$ is, for some α, the category of 'α-precanonical' sheaves on the 'α-pretopos' of its 'α-coherent' objects (these notions being defined just as for the case $\alpha = \omega$ except that α-coproducts instead of finite coproducts and α-filtered colimits instead of colimits over directed sets, are considered). The theorem may be briefly stated: 'Every Grothendieck topos has enough Boolean-valued points', since the topoi $\mathfrak{X}$ mentioned are just those of the form sh($\boldsymbol{B}$, $\mathfrak{S}$) for some CBA in $\mathfrak{S}$ (recall that the *atomless* CBA's are the obvious obstruction to the existence of ordinary $\mathfrak{S}$-valued points.)

THEOREM 12 ([1, Vol. II], for the case of $\mathfrak{S}$ = abstract sets; the case stated should present no significant further difficulties). *Let $\mathfrak{S}$ be a topos with natural numbers object and satisfying the axiom of choice. Then for any topos $\mathfrak{Y}$ bounded over $\mathfrak{S}$ there exists a topos $\mathfrak{X}$ over $\mathfrak{S}$ which also satisfies the axiom of choice and a continuous map $\mathfrak{X} \to \mathfrak{Y}$ over $\mathfrak{S}$ whose inverse-image functor is faithful.*

The difficult part of this theorem is to show that there is such a 'very dominant' map $\mathfrak{Y}' \to \mathfrak{Y}$ with $\mathfrak{Y}'$ generated by subobjects of 1 (relative to $\mathfrak{S}$) since then known methods (e.g. [6]) for embedding Heyting algebras in Boolean algebras give easily an $\mathfrak{X}$ as described with a very dominant $\mathfrak{X} \to \mathfrak{Y}'$. I do not know if the choice of $\mathfrak{X} \to \mathfrak{Y}$ can be made reasonably canonical (in the way that, as remarked above, the choice of the sheaf representation for $\mathfrak{Y}$ can be. Of course, if $\mathfrak{Y}$ happens to have a sufficient set K of *ordinary* points, we can take $\mathfrak{X} = \mathfrak{S}/K$ (corresponding to the Boolean algebra of subsets of K).

Recalling that [21] for a continuous map $\mathfrak{X} \to \mathfrak{Y}$ of topoi with faithful inverse-image functor, $\mathfrak{Y}$ is equivalent to the category of co-algebras for the induced comonad on $\mathfrak{X}$, we may contrast the sheaf representation of $\mathfrak{Y}$ with this one by noting that there the motion is explained internally, whereas here it is analysed in terms of random motions conditioned externally by the comonad which we have discovered in the process of analysis.

However, even when there are points, there are constructions for which using that information may lead to unnecessary complication, leaving it unclear how to generalize the construction to other topoi. For example the construction of the spectrum of a commutative ring using primes, Zariski sets of primes, localization at primes to get the stalks, then putting these together to obtain a sheaf of local rings is quite complicated, yet the end result can be obtained more directly without using such internal points, which moreover need not exist even for a base topos such as the space of all rings which has plenty of external points. The desired end result of the construction is uniquely determined by the following universal mapping property [9]; we are given a commutative ring A in a topos $\mathfrak{S}$ and want to construct a topos Spec(A) over $\mathfrak{S}$ containing a *local* ring object $A^{\#}$ which is an A-algebra, such that for any topos $\mathfrak{X}$ over $\mathfrak{S}$ with a local ring B in it which is an A-algebra there is a unique pair u, φ consisting of a continuous map $\mathfrak{X} \xrightarrow{u} \mathrm{Spec}(A)$ over $\mathfrak{S}$ and a *local* homomorphism $A^{\#} \xrightarrow{\varphi} B$ of A-algebras. Note that the problem of finding a local ring to which A maps universally is unreasonable if we remain within a fixed set theory $\mathfrak{S}$; we have to allow spreading out the domain of variation of the set theory in order to hope to solve it. Joyal's method of solving this problem is to consider another universal problem internal to $\mathfrak{S}$ which can be solved and to show that the solution of the internal problem leads to a solution of the original global spectrum problem; this internal problem may be thought of as an idealization of a practical problem of the following kind: Suppose that through investigation we have sufficient knowledge of some variable quantities such as air pressure, distance to Rome, number of cattle, iron concentration, water velocity, etc. to know the relations between their sums, products, etc. which knowledge we regard as determining a commutative ring A. The problem is to draw a map of the domain of variation of these quantities showing the regions D_f where each quantity f exists (i.e., does not vanish) so that in particular we must determine all the inclusion relations between these regions so that we can picture how they overlap; if we consider the system of all finite unions of such regions, our problem is to determine a certain *distributive lattice* object $L(A)$ in $\mathfrak{S}$ and a morphism $A \xrightarrow{D} L(A)$ satisfying the conditions

$$D_1 = 1,$$
$$D_{fg} = D_f \wedge D_g,$$
$$D_{f+g} \subseteq D_f \vee D_g,$$

and no more in the sense that any morphism $A \xrightarrow{d} L$ to any distributive lattice object L which satisfies the three conditions will factor uniquely through D by means of a lattice homomorphism $L(A) \to L$. If we consider the free $\vee$-semilattice $\mathcal{P}_\omega(A)$ generated by the 'set' A, the multiplication in A extends to an associative multiplication in $\mathcal{P}_\omega(A)$ distributive with respect to $\vee$, so we can define $L(A) = \mathcal{P}_\omega(A)/\equiv$ as the quotient modulo the semiring congruence generated by the third condition above and the condition that the multiplication should be idempotent. Since this whole discussion has been untainted by the Greek notion of point, it applies equally well in any topos $\mathfrak{S}$ (for example we may imagine that the sets of $\mathfrak{S}$ and so in particular the quantities in A are already varying in the domain of time, and hence so is the geometric picture $L(A)$). Spec(A) can then be constructed as the precanonical (i.e., coverings are finite sups) $\mathfrak{S}$-valued sheaves on $L(A)$ and $A^{\#}$ as the presheaf $f \rightsquigarrow A[f^{-1}]$.

In the case where $\mathfrak{S}$ is the category of abstract sets, the theorem of Deligne allows us to state a theorem of Hochster in the following form, which is presumably also true for any topos having a natural-numbers object and satisfying the axiom of choice.

THEOREM 13 (Hochster). *The topoi over $\mathfrak{S}$ which are of the form* Spec(A) *for some commutative ring A in $\mathfrak{S}$ are precisely those which are coherent and generated by subobjects of* 1 *(over $\mathfrak{S}$). (These same topoi appear as the underlying topoi of these schemes which may be covered by a finite number of affine opens).*

Coherent topoi which are not generated by subobjects of 1 (i.e., which are not of the form: sheaves on a coherent topological space[4] also occur

[4] Contrary to what is stated on [1, Vol. II, exposé 6, p. 1], a separated coherent space is just a Boolean space, not necessarily a finite one.

in algebraic geometry (and thus led to the creation of topos theory) for example the sheaf $A^{\#}$ of local rings on Spec(A) is classified by a continuous map Spec(A) $\to \mathfrak{Z}$ where $\mathfrak{Z}$ is the Zeriski topos, and any algebraic group is determined by a continuous map $\mathfrak{Z} \to \mathfrak{S}^{G}$ to the coherent topos $\mathfrak{S}^{G}$ of presheaves on the category G of finitely presented groups. Recalling the geometric logic of Joyal–Reyes discussed above, we could claim

Algebraic Geometry = Geometric Logic

since each can be transformed into the other on the basis of the fact that both are the study of continuous maps between coherent topoi. This claim metaphysically ignores the dominating aspect in algebraic geometry of calculations in *linear* algebra; on the other hand I have maintained elsewhere that logic should be regarded as including the formalism of closed categories (not only as a particular theory, but as an extended 'pure' logic), whereas the form of the linear algebra calculations in geometry *is* that of (abelian and) closed categories.

References

[1] M. Artin, A. Grothendieck and J. L. Verdier, *Théorie des Topos et Cohomologie Étale des Schémas*, Vols. 1 and 2, Lecture Notes in Mathematics, Vols. 269 and 270 (Springer, Berlin, 1972).

[2] M. Barr, *Exact categories*, Lecture Notes in Mathematics, Vol. 236 (Springer, Berlin, 1971).

[3] M. Barr, Toposes without Points, *Journal of Pure and Applied Algebra*, 5 (1974) 265–280.

[4] J. P. Cleave, Cauchy, convergence and continuity, *The British Journal for the Philosophy of Science* **22** (1971) 27–37.

[5] R. Diaconescu, Change of base for some toposes, Thesis, Dalhousie University, Halifax, U.S. (1973).

[6] P. Freyd, Aspects of topoi, *Bulletin of the Australian Mathematical Society* **7** (1972) 1–76.

[7] J. Giraud, *Cohomologie non Abélienne* (Springer, Berlin, 1971).

[8] J. Giraud, *Classifying Topos*, Lecture Notes in Mathematics, Vol. 274 (Springer, Berlin, 1972).

[9] M. Hakim, *Topos annelés et schémas relatifs* (Springer, Berlin, 1972).

[10] M. Hochster, Prime ideal Structure in commutative rings, *Transactions of the American Mathematical Society* **142** (1969) 43–60.

[11] A. Kock and Chr. Juul Mikkelson, Topos-theoretic factorization of non-standard extensions, in: *Proceedings of the Victoria Symposium*, Lecture Notes in Mathematics, Vol. 369 (Springer, Berlin).

[12] A. Kock and G. C. Wraith, Elementary toposes, Aarhus Universitet, Lecture Notes No. 30 (1971).

[13] F. W. Lawvere, Quantifiers and sheaves, *Actes du Congrès International des Mathematiques* (1970) tome 1, p. 329.
[14] F. W. Lawvere, Introduction, in: *Toposes, Algebraic Geometry and Logic*, Lecture Notes in Mathematics, Vol. 274 (Springer, Berlin, 1972).
[15] F. W. Lawvere, Metric spaces, closed categories and generalized logic, *Rendiconti Seminario Matematico e Fisico di Milano* **43** (1973) to appear.
[16] S. Mac Lane, *Categories for the Working Mathematician* (Springer, Berlin, 1971).
[17] W. Mitchell, Boolean topoi and the theory of sets, *Journal of Pure and Applied Algebra* **2** (1972) 261–274.
[18] W. Mitchell, On topoi as closed categories, *Journal of Pure and Applied Algebra* **3** (1973) 133–139.
[19] R. Paré, Colimits in topoi, *Bulletin of the American Mathematical Society* **80** (1974) 556–561.
[20] G. Reyes, From sheaves to logic, to appear.
[21] M. Tierney, Axiomatic sheaf theory, C.I.M.E Lectures, Varenna (Sept. 1971).
[22] G. Wraith, Object classifiers, unpublished.