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INTRODUCTION

by

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The program of investigating the connections between algebraic geometry and "intuitionistic" logic under the guidance of the form of objective dialectics known as category theory was discussed and moved forward at a conference in January 1971 at Halifax, Nova Scotia where seventy mathematicians representing several fields took part. Some of the lectures delivered are reflected in the seven articles in this volume and in the present introduction.

Our own hopes in the success of the above general program were strengthened by initial progress in carrying out a more special program which will be outlined in this introduction. This is the development on the basis of elementary (first-order) axioms of a theory of "toposes" just good enough to be applicable not only to sheaf theory, algebraic spaces, global spectrum, etc. as originally envisaged by Grothendieck, Giraud, Verdier, and Hakim but also to Kripke semantics, abstract proof theory, and the Cohen-Scott-Solovay method for obtaining independence results in set theory. At Rome and Oberwolfach meetings in Spring 1969 I had discussed this program and proposed a set of axioms (essentially theorems 1 and 2 below) which were then shown during my 1969 - 70 collaboration with Myles Tierney to be adéquate for all the usual exactness properties of toposes as well as for the construction of sheaf categories and the proof that they are again toposes. That part of our joint work dealing with the continuum hypothesis is detailed in Tierney's article in this volume. We also simplified the original axioms, a process which has been carried further more recently by Chris Juul-Mikkelsen. The proof of the exactness of the associated-sheaf functor has recently been simplified by Peter Freyd, who has also made interesting contributions to the relationship between right-exactness and number theory in a topos. Dana Scott has pointed out that Dedekind-cut sense of "analysis in a topos" reduces to his model for intuitionistic analysis in the "classical" case of sheaves over a non-trivial topological space.

We now understand by a topos any category $\underline{E}$ which is cartesian closed and has a subobject-representor. Thus a topos has a terminal object 1 and cartesian product and exponential functors determined by the adjointness relations

$$\frac{X \longrightarrow Y_1 \times Y_2}{X \longrightarrow Y_1, X \longrightarrow Y_2} \qquad\qquad \frac{X \longrightarrow Y^A}{A \times X \longrightarrow Y}$$

as well as a "truth-value" object Ω satisfying the adjointness relation

$$\frac{X \longrightarrow \Omega}{? \rightarrowtail X}$$

where $? \rightarrowtail X$ refers to an arbitrary equivalence class of monomorphisms into X (i.e. an arbitrary subobject of X). More exactly of course the natural bijections indicated by horizontal lines above are mediated by unities

$$X \xrightarrow{\delta} X \times X \qquad\qquad X \xrightarrow{\lambda} (A \times X)^A$$

$$Y_1 \times Y_2 \xrightarrow{\pi i} Y_i \qquad\qquad A \times Y^A \xrightarrow{\varepsilon} Y$$

in the case of the cartesian closed structure, and by

$$1 \xrightarrow{\text{true}} \Omega$$

in the case of the subobject representor. To explain more precisely the working of the latter, regard any morphism $A \xrightarrow{x} X$ as an element of X "defined over A" (this has its usual sense in the case of algebraic geometry) and for any monomorphism $S \overset{m}{\rightarrowtail} X$ say that

$x \; \varepsilon \; m$ iff there exists $\bar{x}$ such that $x = \bar{x}\, m$

$$\begin{array}{ccc} A & \overset{\bar{x}}{\dashrightarrow} & S \\ & {\scriptstyle x}\searrow \quad \swarrow{\scriptstyle m} & \\ & X & \end{array}$$

Further, write true_A for the composite (constant) morphism $A \longrightarrow 1 \xrightarrow{\text{true}} \Omega$. Then the determining property of Ω is as follows: Given any "propositional function" $X \xrightarrow{\phi} \Omega$ there is a monomorphism $\{X|\phi\}$ with codomain X such that for any $A \xrightarrow{x} X$

$$x \; \varepsilon \; \{X|\phi\} \text{ iff } x\phi = \text{true}_A$$

and conversely every monomorphism with codomain X has a unique "characteristic function" ϕ . (Anders Kock has shown that in fact it suffices to assume the existence of Y^A for the case $Y = \Omega$.)

Briefly we may say that the notion of topos summarizes in objective categorical form the essence of "higher-order logic" (we will explain below how the logical operators become morphisms in a topos) with no axiom of extensionality. This amounts to a natural and useful generalization of set theory to the consideration of "sets which internally develop". In a basic example of algebraic geometry, the development may be viewed as taking place along a parameter which varies over "rings of definition"; in a basic example from intuitionistic logic, the parameter is interpreted as varying over "stages of knowledge". To illustrate we further describe an example and four classes of examples.

The most "abstract" topos is the familiar category $\underline{S}$ of abstract sets and mappings in which, so to speak, the development has been frozen so that morphisms $X \longrightarrow Y$ are entirely determined by what they do to "global" or "external" elements of X , i.e. elements $1 \longrightarrow X$ defined over the terminal object 1. Here of course Y^A is an abstract set which precisely indexes the morphisms $A \longrightarrow Y$ and Ω is a two-element abstract set. There being no development going on in the objects of $\underline{S}$, there is nothing to obstruct the existence of choice functions, and indeed the axiom of choice in a certain sense characterizes models of set theory among toposes. More exactly, Radu Diaconescu has shown that any topos in which epimorphisms split is also generated by the subobjects of 1 and has $\Omega \cong 1 + 1$ (coproduct) and is hence (in view of the results discussed below) a "Boolean-valued model for the elementary theory of the category of sets" if it satisfies an axiom of infinity.

The first class of toposes to be studied as categories was the class of $\underline{E}$ of the form $\underline{E}$ = all $\underline{S}$-valued sheaves on some topological space. In such an example our axioms are verified in terms of the section functor Γ as follows

$$\Gamma(U, Y^X) = \mathrm{Hom}\,(X|U, Y|U)$$

for all sheaves X and Y and all open sets U , and

$$\Gamma(U, \Omega) = \text{Set of all open subsets of } U$$

A related class of toposes are those of the form $\underline{S}^{\underline{P}}$ where $\underline{P}$ is a poset. Here an object X may be analyzed as a family of abstract sets indexed by the elements of $\underline{P}$ and equipped with transition mappings $X_p \longrightarrow X_q$ for $p \leq q$, satisfying the conditions that the transition mapping $X_p \longrightarrow X_p$ is the identity and that the diagram

$$\begin{array}{ccc} X_p & \longrightarrow & X_q \\ & \searrow \quad \swarrow & \\ & X_r & \end{array}$$

of transition mappings commutes whenever $p \leq q \leq r$. A morphism $X \xrightarrow{f} Y$ is any family $X_p \xrightarrow{fp} Y_p$ of mappings which commutes with the transition mappings

$$\begin{array}{ccc} X_p & \xrightarrow{f_p} & X_p \\ \downarrow & & \downarrow \\ X_q & \xrightarrow[f_q]{} & Y_q \end{array} \qquad \text{Whenever } p \leq q \ .$$

Such a category $\underline{S}^{\underline{P}}$ is a topos, with

$(Y^X)_p \cong$ set of all families f_q as above, except defined only for those q with $p \leq q$,

and

$\Omega_p \cong$ set of all those subsets S of $\underline{P}$ which satisfy $q \in S \Rightarrow p \leq q$ and
$q \in S$ and $q \leq r \Rightarrow r \in S$

with the transition mappings $(Y^X)_p \longrightarrow (Y^X)_q$ and $\Omega_p \longrightarrow \Omega_q$ given by restricting. By considering

$$\bar{X}_p = \sum_{s \leq p} X_s$$

we see that any object X is the quotient of an $\bar{X}$ which "increases" (in the sense that the transitions are monomorphisms) modulo an equivalence relation $E_p \subset \bar{X}_p \times \bar{X}_p$ which also increases; this shows the relationship between toposes of the form $\underline{S}^{\underline{P}}$ and the usual model theory for "intuitionistic" logic - namely we need only take account of "equality" in the latter to reduce it to the former. Note that toposes of the form $\underline{S}^{\underline{P}}$ share with toposes of sheaves on topological spaces the property of having non-Boolean internal logic except in the most trivial cases.

Even before sheaf theory or intuitionistic logic mathematicians considered permutation representations of groups, and these give rise also to a distinctive class of toposes. Slightly more generally, let $\underline{G}$ be any Brandt groupoid and let $\underline{S}^{\underline{G}}$ be the category of representations of $\underline{G}$ in abstract sets, with equivariant maps as morphisms. Then

$$(Y^X)_p = Y_p^{X_p} \text{ (all mappings)}$$

for each identity p of $\underline{G}$, with the obvious action $f^g = g^{-1}fg$,

and

$$\Omega_p = \text{two element set}$$

with trivial action. In a sense which can be made precise, these are the only toposes which can be defined over $\underline{S}$ in a way which preserves so strictly the topos structure.

By the way of contrast let $\underline{M}$ be any monoid which is not a group (and, for uniqueness of presentation, assume it has no non-identity idempotents). Then in the topos $\underline{S}^{\underline{M}}$ of $\underline{M}$-sets, $\underline{M}$ acting on itself is a canonical generator and we have that

Y^X = set of all equivariant maps $\underline{M} \times X \longrightarrow Y$

Ω = set of all left ideals of $\underline{M}$

both with a natural action of $\underline{M}$.

The three classes just described are of course subsumed under the more general class of toposes having the form of a functor category $\underline{S}^{\underline{C}^{op}}$ where $\underline{C}$ is any small category. Now one of the important features of the theory of toposes is that a great many constructions can be <u>relativised</u> through replacing $\underline{S}$ by an arbitrary base topos $\underline{E}$, and the functor category construction is one of these. For one thing, significance of the condition that a category $\underline{C}$ is "small" is that its "set" of objects and "set" of morphisms have the nature of objects in the base topos and that its domain, codomain and composition operations have the nature of morphisms in the base topos. For another thing, we have from the topology of fiber bundles the idea that a really internal "family" of objects (for example the family of values of a functor) indexed by an object C_o is simply a morphism $X \longrightarrow C_o$, and this idea is if anything even more sensible in a topos. These two observations can be used to define the notion of category $\underline{C}$ <u>in</u> $\underline{E}$ and to define the category of internal $\underline{E}$-valued functors on $\underline{C}$ for any category $\underline{E}$ with finite limits. A topos E has finite limits, which we can prove by constructing either equalizers or intersections of subobjects: if we denote by θ_Y the characteristic function of the diagonal monomorphism $Y \longrightarrow Y \times Y$, then for any pair f_1, f_2 of morphisms $X \longrightarrow Y$, the composite

$$X \xrightarrow{\langle f_1, f_2 \rangle} Y \times Y \xrightarrow{\theta_Y} \Omega$$

is the characteristic function of the equalizer $\{X | f_1 \, \theta_Y f_2\}$ of f_1 with f_2 ; or if we denote by

$$\Omega \times \Omega \xrightarrow{\wedge} \Omega$$

the characteristic function of $1 \xrightarrow{\langle true, true \rangle} \Omega \times \Omega$, we can, given two subobjects of X with characteristic functions ϕ_1 and ϕ_2 , obtain their intersection as $\{X | \phi_1 \wedge \phi_2\}$.

The fact that the internally-defined functor categories (including the special case $\underline{E}/_{\underline{X}}$) are again toposes, as well as the usual exactness properties of toposes such as pullbacks are exact, pushouts of monos are monos, etc. follow from

<u>Theorem 1</u>. For any morphism $X \xrightarrow{f} Y$ in a topos $\underline{E}$, the functor

$$\underline{E}/_X \xleftarrow[f^*]{} \underline{E}/_Y$$

obtained by pulling back along f has a right adjoint Π_f (as well as the obvious left adjoint Σ_f which is just composition with f). As special cases we have, in addition to the exponentiation in $\underline{E}/_Y$, the partial-morphism representor

$$\tilde{B} = \sum_{\Omega \to 1} \; \prod_{true} B$$

for any object B of $\underline{E}$ (which satisfies

$$\frac{A \longrightarrow \tilde{B}}{A \longleftarrow\!\!\prec \; ? \longrightarrow B}$$

or in other words is the right adjoint to the inclusion of $\underline{E}$ into the category with the same objects but with arbitrary <u>partially-defined</u> morphisms) as well as the operations

$$\Omega^X \xrightarrow{\forall_X} \Omega \qquad \Omega \times \Omega \xrightarrow{\Rightarrow} \Omega$$

of universal quantification over X and implication which may alternatively be defined as the characteristic maps of the name of $true_X$ and of the equalizer of conjunction with the first projection, respectively. Other forms of universal quantification are, for any $X \xrightarrow{f} Y$, a right adjoint

$$\Omega^X \xrightarrow{\forall_f} \Omega^Y$$

with respect to the natural order of Ω for the operation of composing with f, as well as for any X, an operation

$$\Omega^{\Omega^X} \xrightarrow{\bigcap_X} \Omega^X$$

of infinite intersection which forms the "multiplication" part of a triple (dual standard construction, monad) whose functor part is $f \rightsquigarrow \forall_f$ and whose unit part is the singleton map

$$X \xrightarrow[\{\ \}_X]{} \Omega^X$$

(which is just the exponential adjoint of θ_X)

It was with use of universal quantification that Chris Juul-Mikkelsen proved the following

<u>Theorem 2</u>. In a topos there exist a strict initial object 0, union of any two subobjects of any object, disjoint sum of any two objects, image factorization of any morphism into epi and mono, equivalence relation generated by any pair $X \rightrightarrows Y$ of morphisms, and coequalizer of any such pair of morphisms.

It follows from theorems 1 and 2 that all epis are coequalizers and that equivalence relations are universal-effective. Moreover the image factorization gives use to various forms

$$\Omega^X \xrightarrow{\exists_X} \Omega \quad , \quad \Omega^X \xrightarrow{\exists_f} \Omega^Y \quad , \quad \Omega^{\Omega^X} \xrightarrow{\bigcup_X} \Omega^X$$

of existential quantification which satisfy appropriate formal relations (rules of inference) but typically "mean actual existence only locally".

There are at least two forms of the idea of a property holding "locally" in a topos. One is intrinsic, and reflects the idea that any epimorphism $S \longrightarrow 1$ is a covering of $\underline{E}$: thus for example a diagram in $\underline{E}$ is said to locally satisfy some property expressed in the language of toposes if there exists S with $S \longrightarrow 1$ epic such that when the diagram is pulled back to $\underline{E}/S$ it has the property in the sense of the topos $\underline{E}/S$. The other notion is with respect to a given $\Omega \xrightarrow{j} \Omega$ which may be thought of as a modal operator to be read "it is j-locally the case that .." and which satisfies the axioms below which in particular mean that j is equivalent to a Grothendieck topology on $\underline{C}$ in the case of a topos of the form $\underline{S}^{\underline{C}^{op}}$. At the Rome and Overwolfach meetings I had pointed out that the usual notion of a Grothendieck topology is equivalent to a single such morphism j ; Tierney showed that the appropriate axioms on j are simply that $jj = j$ and j preserves finite conjunctions.* A subobject $X' \rightarrowtail X$ with characteristic function $X \xrightarrow{\phi} \Omega$ is said to be j-dense if ϕ is j-locally true i.e. if $\phi\ j = \mathrm{true}_X$. The relationships between the two notions of localness arise from the fact that in a subcategory $\underline{E}_j$ of $\underline{E}$ called the category of j-sheaves, a morphism is an epimorphism in the sense of $\underline{E}_j$ iff its image in the sense of $\underline{E}$ is j-dense. By definition, an object Y of $\underline{E}$ is a j-sheaf iff for every j-dense monomorphism $X' \rightarrowtail X$, every morphism $X' \longrightarrow Y$ can be uniquely extended to a morphism $X \longrightarrow Y$. If Y is a j-sheaf and X is any object of $\underline{E}$, Y^X is again a j-sheaf ; thus the full subcategory $\underline{E}_j$ is cartesian closed. Moreover the image Ω_j of j is a sheaf which is a subobject representor for $\underline{E}_j$; thus the category of j-sheaves is again a topos. In sheaf theory an important construction is the associated-sheaf- functor, a left adjoint to the inclusion functor $\underline{E}_j \rightarrowtail \underline{E}$ which is usually

* including the empty conjunction - true.

constructed by a two-step infinite direct-limit procedure. In the absence of (external) infinite direct limits in the axiomatic setting, I found quite another two-step procedure to construct this adjoint: Given X, consider first the image of the canonical map $X \longrightarrow \Omega^X \longrightarrow \Omega_j{}^X$, then form the j-closure of the resulting sub-object of the sheaf $\Omega_j{}^X$ —— this closure is the associated sheaf of X. It is easy to see that this associated-sheaf functor preserves products; the important fact that it preserves all finite inverse limite (i.e. that it is left exact) was proved by Tìerney using a calculus-of-fractions argument. More recently Freyd has proved the exactness using the facts that every topos has enough injectives, that every injective of $\underline{E}_j$ is injective in $\underline{E}$, and the following

<u>Lemma</u> A diagram

$$\begin{array}{ccc} A & \rightarrowtail & A_1 \\ \downarrow & & \downarrow \\ A_2 & \rightarrowtail & X \end{array}$$

of monomorphisms in a topos is an intersection iff for every injective E, a pair of maps $A_i \longrightarrow E$ has a common extension to X iff it has a common restriction to A.

The inclusions $\underline{E}_j \rightarrowtail \underline{E}$ constitute precisely the full and faithful case of <u>geometrical morphism</u> $\underline{F} \longrightarrow \underline{E}$ between toposes, which means any functor having an exact left adjoint. Another ("surjective") case of geometrical morphism is one for which the adjoint reflects isomorphisms - these are determined by a left exact cotriple (standard construction) on the domain $\underline{F}$. Moreover every geometrical morphism can be uniquely factored into two, the first of which is "surjective" in the sense just described and the second of which is full and faithful. This "image topos" construction applied to a "sections" functor (with "stalks" adjoint)

$$\underline{S}/X \longrightarrow \underline{S}^{\underline{B}^{op}}$$

arising from a topology basis $\underline{B}$ on a set X, gives the usual sheaf category. The latter construction can be relativised, replacing the category of sets by an arbitrary topos.

Carrying out logic, algebra, and analysis within a topos usually requires the axiom of infinity, i.e. the existence of the free unary algebra $N \circlearrowleft s$ on 1 generator. Chris Juul-Mikkelsen has shown in detail that this is equivalent to the existence of free monoid objects, and Peter Freyd has shown that $N \circlearrowleft s$ is characterized by being a fixed point $1 + N \xrightarrow{\approx} N$ and by being the least such in the sense that

$$N \underset{id}{\overset{s}{\rightrightarrows}} N \longrightarrow 1 \text{ is a coequalizer}$$

Over a base topos $\underline{S}$ with N there are two important toposes which should be investigated in more detail. One is the category $\underline{A}$ of sheaves on the product space N^N, which has a nice topology basis; besides its importance for intuitionistic analysis, $\underline{A}$ has the property that "analytic spaces" are determined by left exact cotriples in it. The other is sort of proof-theoretic version of the Dedekind-cut construction which yields a topos $R(\underline{S})$ whose truth-values are (in the $\underline{S}$-sense) nonnegative real numbers (including ∞, and with the "reverse" ordering which is convenient for setting up metric spaces as "strong categories"); if $\underline{Q}$ denotes the poset of non-negative rationals, $R(\underline{S})$ is the subcategory of $\underline{S}^Q$ consisting of those X for which $X_q = \varprojlim_{r>q} X_r$. Using Brian Day's theory of convolution, one can extend the usual addition and truncated subtraction of reals to get a (non-cartesian) closed structure on the whole topos $R(\underline{S})$. It should be useful to work out some aspects of differential topology, infinite-dimensional group representations etc. in this setting i.e. "analysis in a topos" making use of the interplay between the external view, to the effect that a topos is a generalized space, and the internal view, to the effect that a topos is a relativized set theory.

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