Ordinal Sums and Equational Doctrines

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Our purpose is to describe some examples and to suggest some directions for the study of categories with equational structure. To equip a category $\mathbf{A}$ with such a structure means roughly to give certain "$\mathbf{C}$-tuples of $\mathbf{D}$-ary operations"

$$\mathbf{A}^{\mathbf{D}} \xrightarrow{\vartheta} \mathbf{A}^{\mathbf{C}}$$

for various categories $\mathbf{D}$ and $\mathbf{C}$, in other words, "operations" in general operate (functorially or naturally) on diagrams in $\mathbf{A}$, not only on n-tuples, and may be subjected to equations involving both composition of natural transformations and Godement multiplication of natural transformations and functors. By an equational doctrine we mean an invariant form of a system of indices and conditions which specifies a particular species of structure of the general type just described. Thus equational doctrines bear roughly the same relation to the category of categories which algebraic theories bear to the category of sets. Further development will no doubt require contravariant operations (to account for closed categories) and "weak algebras" (to allow for even the basic triple axioms holding "up to isomorphism"), but in this article we limit ourselves to strong standard constructions in the category of categories.

Thus, for us an *equational doctrine* will consist of the following data:

1) a rule $\mathscr{D}$ which assigns to every category $\mathbf{B}$ another category $\mathbf{B}\mathscr{D}$ and to every pair of categories $\mathbf{B}$ and $\mathbf{A}$, a functor

$$\mathbf{A}^{\mathbf{B}} \xrightarrow{\mathscr{D}} (\mathbf{A}\mathscr{D})^{(\mathbf{B}\mathscr{D})}$$

2) a rule η which assigns to every category $\mathbf{B}$ a functor

$$\mathbf{B} \xrightarrow{\mathbf{B}\eta} \mathbf{B}\mathscr{D}$$

3) a rule μ which assigns to every category $\mathbf{B}$ a functor

$$(\mathbf{B}\mathscr{D})\mathscr{D} \xrightarrow{\mathbf{B}\mu} \mathbf{B}\mathscr{D}$$

These data are subject to seven axioms, expressing that $\mathscr{D}$ is strongly functorial, η, μ strongly natural, and that together they form a standard construction (= monad = triple). For example, part of the functoriality of $\mathscr{D}$ is expressed by the commutativity of

$$\begin{array}{ccc}
\mathbf{B}^{\mathbf{C}} \times \mathbf{A}^{\mathbf{B}} & \xrightarrow{\mathscr{D}\times\mathscr{D}} & (\mathbf{B}\mathscr{D})^{(\mathbf{C}\mathscr{D})} \times (\mathbf{A}\mathscr{D})^{(\mathbf{B}\mathscr{D})} \\
{\scriptstyle \text{compn.}}\big\downarrow & & \big\downarrow{\scriptstyle \text{compn.}} \\
\mathbf{A}^{\mathbf{C}} & \xrightarrow[\mathscr{D}]{} & (\mathbf{A}\mathscr{D})^{(\mathbf{C}\mathscr{D})}
\end{array}$$

while the naturality of μ is expressed by the commutativity of

$$\begin{CD}
\mathbf{A}^{\mathbf{B}} @>{\mathscr{D}^2}>> (\mathbf{A}\mathscr{D}\mathscr{D})^{(\mathbf{B}\mathscr{D}\mathscr{D})} \\
@V{\mathscr{D}}VV @VV{(\mathbf{A}\mu)^{(\mathbf{B}\mathscr{D}\mathscr{D})}}V \\
(\mathbf{A}\mathscr{D})^{(\mathbf{B}\mathscr{D})} @>>{(\mathbf{A}\mathscr{D})^{(\mathbf{B}\mu)}}> (\mathbf{A}\mathscr{D})^{(\mathbf{B}\mathscr{D}\mathscr{D})}
\end{CD}$$

and the associativity of μ by the commutativity of

$$\begin{CD}
\mathbf{B}\mathscr{D}\mathscr{D}\mathscr{D} @>{(\mathbf{B}\mathscr{D})\mu}>> (\mathbf{B}\mathscr{D})\mathscr{D} \\
@V{\ulcorner \mathbf{B}\mu \urcorner \mathscr{D}}VV @VV{\mathbf{B}\mu}V \\
(\mathbf{B}\mathscr{D})\mathscr{D} @>>{\mathbf{B}\mu}> \mathbf{B}\mathscr{D}
\end{CD}$$

In the last diagram the left column denotes the value of the functor

$$(\mathbf{B}\mathscr{D})^{(\mathbf{B}\mathscr{D}\mathscr{D})} \xrightarrow{\mathscr{D}} (\mathbf{B}\mathscr{D})\mathscr{D}^{(\mathbf{B}\mathscr{D}\mathscr{D})\mathscr{D}}$$

at the object $\ulcorner \mathbf{B}\mu \urcorner$ of its domain which corresponds to the functor $\mathbf{B}\mu$.

An *algebra* (sometimes called a "theory") over the given doctrine means a category $\mathbf{A}$ with a functor $\mathbf{A}\mathscr{D} \xrightarrow{\alpha} \mathbf{A}$ subject to the usual two conditions. Homomorphisms between algebras are also defined as usual, although probably "weak" homomorphisms will have to be considered later too.

For examples of doctrines, consider any category $\mathbf{D}$ and let $\mathscr{D}\colon \mathbf{B} \mapsto \mathbf{B}^{\mathbf{D}}$ with η, μ defined diagonally. Or let $\mathscr{D}\colon \mathbf{B} \mapsto \mathbf{D}^{(\mathbf{D}^{\mathbf{B}})}$ with obvious (though complicated) η, μ. Clearly a *strongly adjoint* equational doctrine is determined by a category $\mathbf{M} = \mathbf{1}\mathscr{D}$ equipped with a stricly associative functorial multiplication $\mathbf{M} \times \mathbf{M} \longrightarrow \mathbf{M}$ with unit.

One of several important operations on doctrines is the formation of the *opposite* doctrine [a]

$$\mathscr{D}^{\mathrm{op}}\colon \mathbf{B} \mapsto ((\mathbf{B}^{\mathrm{op}})\mathscr{D})^{\mathrm{op}}$$

(Note that $(\ )^{\mathrm{op}}$, while covariant, is not a *strong* endofunctor of $\mathbf{Cat}$; however it operates on the strong endofunctors in the manner indicated.)

Denoting by $\mathbf{Cat}^{\mathscr{D}}$ the category of algebras (or theories) over the doctrine $\mathscr{D}$, we define

$$\mathrm{Hom}_{\mathscr{D}}\colon (\mathbf{Cat}^{\mathscr{D}})^{\mathrm{op}} \times \mathbf{Cat}^{\mathscr{D}} \longrightarrow \mathbf{Cat}$$

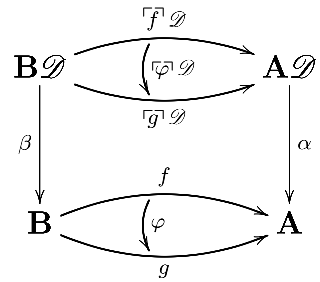
by the *equalizer*

$$\mathrm{Hom}_{\mathscr{D}}(\mathbf{B},\mathbf{A}) \longrightarrow \mathbf{A}^{\mathbf{B}} \rightrightarrows \mathbf{A}^{(\mathbf{B}\mathscr{D})}, \qquad \mathbf{A}^{\beta}\colon \mathbf{A}^{\mathbf{B}} \longrightarrow \mathbf{A}^{(\mathbf{B}\mathscr{D})}, \qquad \mathbf{A}^{\mathbf{B}} \xrightarrow{\mathscr{D}} \mathbf{A}\mathscr{D}^{(\mathbf{B}\mathscr{D})} \xrightarrow{\alpha^{(\mathbf{B},\mathscr{D})}} \mathbf{A}^{(\mathbf{B}\mathscr{D})}$$

[a] Editor's footnote: Here and elsewhere in the original, authors used $\mathbf{A}^*$ rather than $\mathbf{A}^{\mathrm{op}}$, but for this reprint version we have changed it to the current standard notation.

where β, α denote the algebra structures on $\mathbf{B}$, $\mathbf{A}$ respectively. That is if $\mathbf{B} \overset{f}{\underset{g}{\rightrightarrows}} \mathbf{A}$ are two algebra homomorphisms and if $\varphi\colon f \longrightarrow g$ is a natural transformation, then φ is considered to belong to the *category* Hom iff it also satisfies under Godement multiplication the same equation which defines the notion of homomorphism:

$$(\ulcorner\varphi\urcorner\,\mathscr{D})\alpha = \beta\varphi$$



$\mathrm{Hom}_{\mathscr{D}}(\mathbf{B}, \mathbf{A})$ may or may not be a full subcategory of $\mathbf{A}^{\mathbf{B}}$, depending on $\mathscr{D}$.

In particular

$$\mathrm{Hom}_{\mathscr{D}}(\mathbf{1}\mathscr{D}, -)\colon \mathbf{Cat}^{\mathscr{D}} \longrightarrow \mathbf{Cat}$$

is the underlying functor, which has a strong left adjoint together with which it resolves $\mathscr{D}$.

For a given $\mathscr{D}$-algebra $\langle \mathbf{A}, \alpha\rangle$ the functor

$$\mathrm{Hom}_{\mathscr{D}}(-, \mathbf{A})\colon (\mathbf{Cat}^{\mathscr{D}})^{\mathrm{op}} \longrightarrow \mathbf{Cat}$$

might be called "$\mathscr{D}$-semantics with values in $\mathbf{A}$". It has a strong left adjoint, given by $\mathbf{C} \mapsto \mathbf{A}^{\mathbf{C}}$. (That $\mathbf{A}^{\mathbf{C}}$ is a $\mathscr{D}$-algebra for an abstract category $\mathbf{C}$ and $\mathscr{D}$-algebra $\mathbf{A}$ is seen by noting that

$$\mathbf{C} \xrightarrow{\mathrm{ev}} \mathbf{A}^{(\mathbf{A}^{\mathbf{C}})} \longrightarrow (\mathbf{A}\mathscr{D})^{(\mathbf{A}^{\mathbf{C}})\mathscr{D}}$$

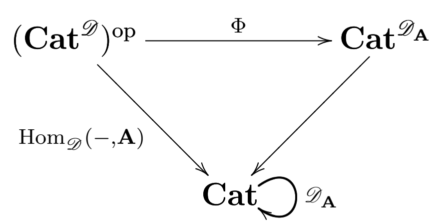
corresponds by symmetry to a functor

$$(\mathbf{A}^{\mathbf{C}})\mathscr{D} \longrightarrow (\mathbf{A}\mathscr{D})^{\mathbf{C}}$$

which when followed by $\alpha^{\mathbf{C}}$ gives the required $\mathscr{D}$-structure on $\mathbf{A}^{\mathbf{C}}$). We thus obtain by composition a new doctrine $\mathscr{D}_{\mathbf{A}}$, the "dual doctrine of $\mathscr{D}$ in the $\mathscr{D}$-algebra $\mathbf{A}$". Explicitly,

$$\mathscr{D}_{\mathbf{A}}\colon \mathbf{C} \mapsto \mathrm{Hom}_{\mathscr{D}}(\mathbf{A}^{\mathbf{C}}, \mathbf{A})$$

The comparison functor Φ in



then has a left adjoint given by

$$\Phi^\vee\colon \mathbf{C} \mapsto \mathrm{Hom}_{\mathscr{D}_\mathbf{A}}(\mathbf{C}, \mathbf{1}\mathscr{D}\Phi)$$

Actually $\mathbf{1}\mathscr{D}\Phi = \mathbf{A}$ as a category, but with the induced $\mathscr{D}_\mathbf{A}$-structure, rather than the given $\mathscr{D}$-structure.

For a trivial example, note that if 1 denotes the identity doctrine, then $\mathrm{Hom}_1(\mathbf{B}, \mathbf{A}) = \mathbf{A}^\mathbf{B}$ and $1_\mathbf{A}\colon \mathbf{C} \mapsto \mathbf{A}^{(\mathbf{A}^\mathbf{C})}$. The dual $1_\mathbf{A}$ of the identity doctrine in $\mathbf{A}$ thus might be called the full 2-clone of $\mathbf{A}$; it takes on a somewhat less trivial aspect if we note that giving $\mathbf{A}$ a structure α over any doctrine $\mathscr{D}$ induces a morphism

$$\mathscr{D} \xrightarrow{\widetilde{\alpha}} 1_\mathbf{A}$$

of doctrines, since

$$\mathbf{A}^\mathbf{C} \xrightarrow{\mathscr{D}} \mathbf{A}\mathscr{D}^{\mathbf{C}\mathscr{D}}$$

yields by symmetry a functor which can be composed with $\alpha^{(\mathbf{A}^\mathbf{C})}$. The image $\mathscr{D}/\langle \mathbf{A}, \alpha\rangle$ of $\widetilde{\alpha}$, if it could be defined in general, would then be the doctrine of "$\mathscr{D}$-algebras in which hold all equations valid in $\langle \mathbf{A}, \alpha\rangle$". In a particular case Kock has succeeded in defining such an image doctrine, and put it to good use in the construction of the doctrine of colimits (see below).

For a more problematic example of the dual of a doctrine, let $\mathbf{S}_0$ denote the skeletal category of *finite* sets, and let $[\mathbf{S}_0, \ulcorner\mathbf{B}\urcorner]$ denote the category whose objects are arbitrary

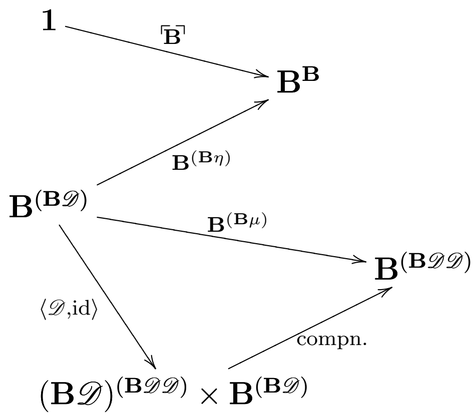
$$n \xrightarrow{B} \mathbf{B}, \quad n \in \mathbf{S}_0$$

and whose morphisms are given by pairs,

$$n \xrightarrow{\sigma} n', \quad B \xrightarrow{b} \sigma B'$$

Then $\mathbf{B} \mapsto [\mathbf{S}_0, \ulcorner\mathbf{B}\urcorner]$ becomes a doctrine by choosing a strictly associative sum operation in $\mathbf{S}_0$ with help of which to define μ. The algebras over the resulting doctrine are arbitrary categories equipped with strictly associative finite coproducts. Algebras over the opposite doctrine $\mathscr{D}$ are then categories equipped with strictly associative finite *products*. By choosing a suitable version(*not* skeletal) of the category $\mathbf{S}$ of small sets, it can be made into a particular algebra $\langle A, \alpha\rangle = \langle \mathbf{S}, \times\rangle$ over $\mathscr{D}$. Then $\mathrm{Hom}_\mathscr{D}(-, \mathbf{S})$ is seen to include by restriction the usual functorial semantics of algebraic theories. Thus in particular *every algebraic category* $\mathbf{C}$ has canonically the structure of a $\mathscr{D}_\mathbf{S}$-algebra, $\mathscr{D}_\mathbf{S}$ denoting the dual doctrine $\mathbf{C} \mapsto \mathrm{Hom}_{\mathrm{prod}}(\mathbf{S}^\mathbf{C}, \mathbf{S})$. The latter doctrine is very rich, having as operations arbitrary $\varprojlim$, directed $\varinjlim$ and probably more (?). Thus if $\mathbf{C}$ is the category of algebras over a small theory, $\mathbf{C}\Phi^\vee = \mathrm{Hom}_{\mathscr{D}_\mathbf{S}}(\mathbf{C}, \mathbf{S})$ must consist of functors which are representable by finitely generated algebras. Thus if one could further see that a sufficient number of coequalizers were among the $\mathscr{D}_\mathbf{S}$ operations (meaning that the representing algebras would have to be projective) we would have a highly natural method of obtaining all the information about an algebraic theory which could possibly be recovered from its category of algebras alone, namely the method of the dual doctrine (which goes back to at least M. H. Stone in the case of sets).

Another construction possible for any doctrine $\mathscr{D}$ is that of $\mathbf{B}_{\mathscr{D}}$, *the category of all possible $\mathscr{D}$-structures on the category* $\mathbf{B}$. It is defined as the $\varprojlim$ of the following finite diagram in **Cat**:



Thus the notion of *morphism* between different $\mathscr{D}$-structures on the same category $\mathbf{B}$ is defined by imposing the same equations on *natural transformations* which are imposed on functors in defining the individual structures. For example, with the appropriate $\mathscr{D} = (\ )\times\mathbf{\Delta}$ defined below, $\mathbf{B}_{\mathscr{D}} = \mathrm{Trip}(\mathbf{B})$ = the usual category of all standard constructions in $\mathbf{B}$. Incidentally, we might call a doctrine $\mathscr{D}$ *categorical* if for any $\mathbf{B}$, any two objects in $\mathbf{B}_{\mathscr{D}}$ are uniquely isomorphic; this would *not* hold for the doctrine of standard constructions, but would for various doctrines of limits or colimits, such as those whose development has been begun by Kock [Kock (1967/68)] ($\mathbf{B}_{\mathscr{D}}$ will of course be $\mathbf{0}$ for many $\mathbf{B}$).

By the *ordinal sum* of two categories we mean the pushout

$$\begin{array}{ccc} \mathbf{A}\times|\mathbf{2}|\times\mathbf{B} & \longrightarrow & \mathbf{A}\times\mathbf{2}\times\mathbf{B} \\ \downarrow & & \downarrow \\ \mathbf{A}+\mathbf{B} & \longrightarrow & \mathbf{A}+_{\mathscr{O}}\mathbf{B} \end{array}$$

in which the left vertical arrow takes $\langle a,i,b\rangle \mapsto a$ if $i=0$, $\mapsto b$ if $i=1$. Thus $\mathbf{A}+_{\mathscr{O}}\mathbf{B}$ may be visualized as $\mathbf{A}+\mathbf{B}$ with exactly one morphism $A\longrightarrow B$ adjoined for every $A\in\mathbf{A}$, $B\in\mathbf{B}$. Actually what we have just defined is the ordinal sum over $\mathbf{2}$; we could also consider the ordinal sum over any category $\mathbf{C}$ of any family $\{\mathbf{A}_C\}$ of categories indexed by the objects of $\mathbf{C}$. For example, with the help of the ordinal sum over $\mathbf{3}$ we see that $+_{\mathscr{O}}$ is an associative bifunctor on **Cat**; it has the empty category $\mathbf{0}$ as neutral object. Also $\mathbf{1}+_{\mathscr{O}}\mathbf{1}=\mathbf{2}$, $\mathbf{1}+_{\mathscr{O}}\mathbf{2}=\mathbf{3}$, etc. One has $\mathbf{1}+_{\mathscr{O}}\omega\cong\omega$ but $\omega+_{\mathscr{O}}\mathbf{1}\ncong\omega$, showing that $+_{\mathscr{O}}$ is *not* commutative; it is not even commutative when applied to *finite* ordinals, if we consider what it does to morphisms.

Now $\mathbf{B} \mapsto \mathbf{1} +_{\mathscr{O}} \mathbf{B}$ may be seen to be the doctrine whose algebras are categories equipped with an initial object, while its opposite doctrine $\mathbf{B} \mapsto \mathbf{B} +_{\mathscr{O}} \mathbf{1}$ is the doctrine of terminal objects.

Consider the category-with-a-strictly-associative-multiplication (denoted by juxtaposition) generated as such by an object T and two morphisms

$$T^2 \xrightarrow{\mu} T \xleftarrow{\eta} 1$$

subject to the three laws familiar from the definition of standard construction. Denote this (finitely-presented!) category with multiplication by $\mathbf{\Delta}$. Clearly then $(\ ) \times \mathbf{\Delta}$ is a doctrine whose algebras are precisely standard constructions. To obtain a concrete representation of $\mathbf{\Delta}$, define a functor

$$\mathbf{\Delta} \longrightarrow \mathbf{Cat}$$

by sending $1 \mapsto \mathbf{0}$, $T \mapsto \mathbf{1}$, $T^2 \mapsto \mathbf{2}$, and noting that since all diagrams ending in $\mathbf{1}$ commute there is a unique extension to a *functor* which takes juxtaposition in $\mathbf{\Delta}$ into *ordinal sum* in $\mathbf{Cat}$. For example, $T\eta, \eta T \mapsto \partial_0, \partial_1$. Clearly the categories which are values of our functor are just all the finite ordinal numbers (including $\mathbf{0}$): we claim that the functor is actually *full* and *faithful*. For suppose

$$\mathbf{n} \xrightarrow{\sigma} \mathbf{m}$$

is any functor (order-preserving map) between finite ordinals. Then

$$\mathbf{m} = \sum_{i \in \mathbf{m}}^{\mathscr{O}} \mathbf{1}$$

and denoting by $\mathbf{n}_i$ the inverse image of i by σ, we actually have that σ itself is an ordinal sum

$$\sigma = \sum_{i \in \mathbf{m}}^{\mathscr{O}} \sigma_i$$

where $\sigma_i \colon \mathbf{n}_i \longrightarrow \mathbf{1}$. Since such σ_i is unique we need only show that $\mathbf{n} \longrightarrow \mathbf{1}$ can be somehow expressed using composition and juxtaposition in terms of T, η, μ. For this define $\mu_n \colon T^n \longrightarrow T$ by

$$\mu_0 = \eta \quad \text{(corresponding to an empty fiber } \mathbf{n}_i\text{)}$$

$$\mu_1 = T \quad \text{(corresponding to a singleton fiber } \mathbf{n}_i\text{)}$$

$$\mu_2 = \mu \quad \text{(corresponding to a two-point fiber } \mathbf{n}_i\text{)}$$

$$\mu_{n+2} = \mu_{n+1} T . \mu$$

Thus $\mu_{n+2} = \mu T^n . \mu T^{n-1} \ldots \mu T . \mu$ and every map is a juxtaposition (ordinal sum) of the μ's. Furthermore any calculation involving the order-preserving maps can be carried out using only the triple laws. Thus $\mathbf{\Delta}$ could also be given the usual (infinite) presentation as a pure category with generators

$$d_i = T^i \eta T^{n-i} \colon T^n \longrightarrow T^{n+1}, \qquad i = 0, \ldots, n$$

$$s_i = T^i \mu T^{n-i} \colon T^{n+2} \longrightarrow T^{n+1}, \qquad i = 0, \ldots, n$$

if desired, although the finite presentation using ordinal sums and the triple laws seems much simpler.

It results in particular that the category $\mathbf{\Delta}$ of finite ordinals (including $\mathbf{0}$) and order-preserving maps carries a canonical standard construction $\mathbf{n} \mapsto \mathbf{1} +_{\mathscr{O}} \mathbf{n}$ (just the restriction of the doctrine of initial objects from **Cat** to $\mathbf{\Delta}$). Denote by $A\mathbf{\Delta}$ the category of algebras for this standard construction, which is easily seen to have as objects all non-zero ordinals and as morphisms the order-preserving maps *which preserve first element*. By construction $A\mathbf{\Delta}$ carries a standard *co*-construction. But it also has a $\mathbf{\Delta}$-structure because it is a *self-dual* category. Namely, since a finite ordinal is a complete category, and since on such a functor preserving initial objects preserves all colimits, we have the isomorphism "taking right adjoints":

$$(A\mathbf{\Delta})^{\mathrm{op}} \xrightarrow[\cong]{\text{adj.}} \mathbf{\Delta} A$$

where $\mathbf{\Delta} A$ denotes the category of maps preserving last element. But now the covariant operation $(\ )^{\mathrm{op}}$ on **Cat** restricts to $\mathbf{\Delta}$ and takes $\mathbf{\Delta} A$ into $A\mathbf{\Delta}$. Thus composing these two processes we obtain the claimed isomorphism

$$(A\mathbf{\Delta})^{\mathrm{op}} \xrightarrow{\cong} A\mathbf{\Delta}$$

and hence a standard construction on $A\mathbf{\Delta}$.

Now let $\mathbf{A}$ be any category equipped with a standard construction T, which we interpret as a category with a given action of $\mathbf{\Delta}$. Then

$$\mathrm{Hom}_{\mathbf{\Delta}}(A\mathbf{\Delta}, \mathbf{A}) \cong \mathbf{A}^{\mathsf{T}}$$

the Eilenberg–Moore category of $\langle \mathbf{A}, \mathsf{T} \rangle$. Since the latter carries canonically an action of $\mathbf{\Delta}^{\mathrm{op}}$, we see that $A\mathbf{\Delta}$ has in another sense the *co-structure* of a standard co-construction, and get an adjoint pair

$$\mathbf{Cat}^{\mathbf{\Delta}} \underset{(-)^{\mathrm{op}}_{\mathbf{\Delta}^*} A\mathbf{\Delta}}{\overset{\mathrm{Hom}_{\mathbf{\Delta}}(A\mathbf{\Delta},-)}{\rightleftarrows}} \mathbf{Cat}^{\mathbf{\Delta}^*}$$

in which the lower assigns to every standard co-construction the associated Kleisli category of free coalgebras. For ease in dealing with these relationships it may be useful to use the following notation for $A\mathbf{\Delta}$, in which A is just a symbol:

$$\begin{array}{ccccccc} & 1+3 & & 1+2 & & 1+1 & & 1+0 \\ \ldots & AT^3 & \overrightarrow{\overrightarrow{\rightarrow}} & AT^2 & \underset{A\mu}{\overset{\xi T}{\rightrightarrows}} & AT & \underset{A\eta}{\overset{\xi}{\rightleftarrows}} & A \end{array}$$

Clearly one can also obtain the *doctrine of adjoint triples*, describing a simultaneous action of $\mathbf{\Delta}$ and $\mathbf{\Delta}^*$ related by given adjunction maps. The writer does not know of a simple concrete representation of the resulting category $\widetilde{\mathbf{\Delta}}$ with strictly associative multiplication. The same could be asked for the doctrine of *Frobenius* standard constructions, determined by the monoid in **Cat** presented as follows

$$1 \underset{\varepsilon}{\overset{\eta}{\rightleftarrows}} T \underset{\mu}{\overset{\delta}{\rightleftarrows}} T^2$$

Triple laws for η, μ and cotriple laws for ε, δ are required to hold, as are the following four equations:

$$\begin{aligned} \delta T.T\mu.T\varepsilon &= \mu \\ T\delta.\mu T.\varepsilon T &= \mu \\ \eta T.\delta T.T\mu &= \delta \\ T\eta.T\delta.\mu T &= \delta \end{aligned}$$

An algebra over this doctrine has an underlying triple and an underlying cotriple whose associated free and cofree functors are *the same*. For example, if G is a *finite group*, then in any abelian category $\mathbf{A}$, $AT = \bigoplus_G A$ has such a structure. The characteristic property from group representation theory actually carries over to the general case: there is a "quadratic form" $\beta = \mu.\varepsilon\colon T^2 \longrightarrow 1$ which is "associative" $T\mu.\beta = \mu T.\beta$ and "non-singular" i. e. there is $\alpha = \eta.\delta\colon 1 \longrightarrow T^2$ quasi-inverse to β (i. e. they are adjunctions for $T \dashv T$.)

In order to construct doctrines whose algebras are categories associatively equipped with colimits, Kock [Kock (1966)] considers categories $\mathbf{Cat}_0$ of categories and functors which are "regular" in the sense that the total category of a fibration belongs to $\mathbf{Cat}_0$ whenever the base and every fiber belong to $\mathbf{Cat}_0$. In order to make $\mathbf{B} \mapsto [\mathbf{Cat}_0, \ulcorner\mathbf{B}\urcorner] = \mathrm{Dir}_{\mathbf{Cat}_0}(\mathbf{B})$ into a strict standard construction in **Cat**, Kock found it necessary to construe $\mathbf{Cat}_0$ as having for each of its objects $\mathbf{C}$ a given well ordering on the set of objects of $\mathbf{C}$ and on each hom set of $\mathbf{C}$. Then with considerable effort he is able to choose a version of the Grothendieck process (taking $\mathbf{C} \xrightarrow{R} \mathbf{Cat}_0$ for $\mathbf{C} \in \mathbf{Cat}_0$ to the associated op-fibration over $\mathbf{C}$ in $\mathbf{Cat}_0$) which gives rise to a *strictly-associative*

$$\mathrm{Dir}_{\mathbf{Cat}_0}(\mathrm{Dir}_{\mathbf{Cat}_0}(\mathbf{B})) \xrightarrow{\mathbf{B}\mu} \mathrm{Dir}_{\mathbf{Cat}_0}(\mathbf{B})$$

One then defines the colimits-over-indexcategories-in-$\mathbf{Cat}_0$ doctrine $\mathfrak{R}$ by

$$\mathfrak{R} = \mathrm{Dir}_{\mathbf{Cat}_0}(-)\Big/\langle \mathbf{S}, \varinjlim \rangle$$

showing first, also with some effort, that there does exist an equivalent version $\mathbf{S}$ of the category of small (relative to $\mathbf{Cat}_0$) sets which can be equipped with a strictly associative $\varinjlim$ i. e. a colimit assignment which is also an algebra structure

$$\mathrm{Dir}_{\mathbf{Cat}_0}(\mathbf{S}) \xrightarrow{\varinjlim} \mathbf{S}$$

for the "precolimit" doctrine $\mathrm{Dir}_{\mathbf{Cat}_0}$.

By choosing the appropriate $\mathbf{Cat}_0$ and by making use of the "opposite" doctrine construction, one then sees that the notions of a category equipped with small $\varinjlim$, finite $\varprojlim$, or countable products, etc, etc, are all essentially doctrinal. Hence presumably, given an understanding of free products, quotients, Kronecker products, distributive laws, etc for doctrines, so are the notions of abelian category, $\mathscr{S}$-topos, ab-topos (the latter two without the usual "small generating set" axiom) also doctrinal. (In order to view, for example, the distributive axiom for topos as a distributive law in the Barr–Beck sense, it may be necessary to generalize the notion of equational doctrine to allow the associative law for μ or α to hold up to isomorphism (?).)

The value of knowing that a notion of category-with-structure is equationally doctrinal should be at least as great as knowing that a category is tripleable over sets. We have at the moment however no intrinsic characterization of those categories enriched over **Cat** which are of the form $\mathbf{Cat}^{\mathscr{D}}$ for some equational doctrine $\mathscr{D}$. However the Freyd Hom-Tensor Calculus [Freyd (1966)] would seem to extend easily from theories over sets to doctrines over **Cat** to give the theorem: any strongly left adjoint functor

$$\mathbf{Cat}^{\mathscr{D}_1} \longrightarrow \mathbf{Cat}^{\mathscr{D}_2}$$

is given by $(-)\otimes_{\mathscr{D}_1}\mathbf{A}$ where $\mathbf{A}$ is a fixed category equipped with a $\mathscr{D}_2$-structure and a $\mathscr{D}_1$-costructure. For example, consider the (doctrinal) notion of **2**-Topos, meaning a partially ordered set with small sups and finite infs which distribute over the sups (morphisms to preserve just the mentioned structure). Then of course the Sierpinski space represents the "open sets" functor

$$\mathrm{Top}^{\mathrm{op}} \longrightarrow \mathbf{2}\text{-Topos}$$

Consider on the other hand the functor

$$\mathscr{S}\text{-Topos} \longrightarrow \mathbf{2}\text{-Topos}$$

which assigns to every $\mathscr{S}$-topos the set of all subobjects of the terminal object; this is represented by the $\mathscr{S}$-topos $\mathbf{E}$ with one generator X subject to $X \xrightarrow{\cong} X\times X$, hence has a strong left adjoint $-\otimes\mathbf{E}$ which, when restricted to $\mathrm{Top}^{\mathrm{op}}$ is just the assignment of the category of sheaves to each space. Or again consider the functor "taking abelian group objects"

$$\mathscr{S}\text{-Topos} \longrightarrow \text{ab-Topos}$$

Since this is $\mathbf{F} \mapsto \mathrm{Hom}_{\mathrm{finprod}}(\mathbf{Z}, \mathbf{F}U)$ where $\mathbf{Z}$ is the category of finitely generated free abelian groups and $\mathbf{F}U$ denotes the category with finite products underlying the topos $\mathbf{F}$, we see that our functor is represented by $\mathbf{A}$ = the relatively free topos over the category $\mathbf{Z}$ with finite products. Hence there is a strong left adjoint $(-)\otimes\mathbf{A}$ which should be useful in studying the extent to which an arbitrary Grothendieck category differs from the abelian sheaves on some $\mathscr{S}$-site.