The Category of Categories as a Foundation for Mathematics*,**

By

F. William Lawvere

In the mathematical development of recent decades one sees clearly the rise of the conviction that the relevant properties of mathematical objects are those which can be stated in terms of their abstract structure rather than in terms of the elements which the objects were thought to be made of. The question thus naturally arises whether one can give a foundation for mathematics which expresses wholeheartedly this conviction concerning what mathematics is about, and in particular in which classes and membership in classes do not play any role. Here by "foundation" we mean a single system of first-order axioms in which all usual mathematical objects can be defined and all their usual properties proved. A foundation of the sort we have in mind would seemingly be much more natural and readily-useable than the classical one when developing such subjects as algebraic topology, functional analysis, model theory of general algebraic systems, etc. Clearly any such foundation would have to reckon with the Eilenberg-MacLane theory of categories and functors. The author believes, in fact, that the most reasonable way to arrive at a foundation meeting these requirements is simply to write down axioms descriptive of properties which the intuitively-conceived category of all categories has until an intuitively-adequate list is attained; that is essentially how the theory described below was arrived at. Various metatheorems should of course then be proved to help justify the feeling of adequacy. The system to be described is an improved version of the one sketched in Chapter 1 of the author's doctoral dissertation [Columbia, 1963].

By the *elementary theory of abstract categories* we mean the notions of formula and theorem defined as follows

0. For any letters x, y, u, A, B the following are formulas

$$\Delta_0(x) = A\,,\quad \Delta_1(x) = B\,,\quad \Gamma(x, y; u)\,,\quad x = y\,.$$

---

* Research partially supported by an NSF-NATO Postdoctoral Fellowship.

** Received September 8, 1965

These are to be read, respectively, "A is the domain of x", "B is the codomain of x", "u is the composition x followed by y", and "x equals y".

1. If Φ and Ψ are formulas, then

$$\begin{aligned}&[\Phi] \text{ and } [\Psi]\\&[\Phi] \text{ or } [\Psi]\\&[\Phi] \Rightarrow [\Psi]\\&\text{not } [\Phi]\end{aligned}$$

are also formulas.

2. If Φ is a formula and x is a letter, then

$$\forall x[\Phi], \quad \exists x[\Phi]$$

are also formulas. These are to be read, as usual, "for every x, Φ" and "there is an x such that Φ", respectively.

3. A string of marks is a *formula* of the elementary theory of abstract categories iff its being so follows from 0, 1, 2 above. Of course we immediately begin to make free use of various ways of abbreviating formulas. The notion of free and bound variables in a formula can now be defined; we mean by a *sentence* any formula with no free variables, i.e. in which every occurence of each letter x is within the scope of a quantifier $\forall x$ or $\exists x$.

The theorems of the elementary theory of abstract categories are all those sentences which can be derived by logical inference from the following axioms (it is understood that Δ_0, Δ_1 are unary function symbols)

Four bookkeeping axioms

$$\Delta_i(\Delta_j(x)) = \Delta_j(x), \quad i, j = 0, 1.$$

$$\begin{aligned}&\Gamma(x, y; u) \text{ and } \Gamma(x, y; u') \Rightarrow u = u',\\&\exists u[\Gamma(x, y; u)] \Longleftrightarrow \Delta_1(x) = \Delta_0(y),\\&\Gamma(x, y; u) \Rightarrow \Delta_0(u) = \Delta_0(x) \text{ and } \Delta_1(u) = \Delta_1(y).\end{aligned}$$

Identity axiom

$$\Gamma(\Delta_0(x), x; x) \text{ and } \Gamma(x, \Delta_1(x); x).$$

Associativity axiom

$$\begin{aligned}&\Gamma(x, y; u) \text{ and } \Gamma(y, z; w) \text{ and}\\&\Gamma(x, w; f) \text{ and } \Gamma(u, z; g) \Rightarrow f = g.\end{aligned}$$

Besides the usual means of abbreviating formulas, the following (as well as others) are special to the elementary theory of abstract categories:

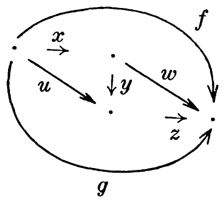
$$A \xrightarrow{f} B \quad \text{means} \quad \Delta_0(f) = A \quad \text{and} \quad \Delta_1(f) = B,$$

$fg = h$ means $\Gamma(f, g; h)$,

$$\begin{array}{c} \overset{f}{A \to B} \\ h \searrow \ \downarrow g \\ C \end{array} \quad \textit{commutes} \text{ means } \quad \begin{array}{l} \Delta_0(f) = \Delta_0(h) = A \quad \text{and} \\ \Delta_1(f) = \Delta_0(g) = B \quad \text{and} \\ \Delta_1(g) = \Delta_1(h) = C \quad \text{and} \\ \Gamma(f, g; h). \end{array}$$

(Notice that we write compositions in the order of the arrows from left to right.)

Commutative diagrams in general are regarded as abbreviated formulas, signifying the usual indicated systems of equations. For example, our statement above of the associativity axiom becomes transparent on contemplating the following commutative diagram, made up of four elementary triangles of the above sort.



Further abbreviated formulas are

Obj(A) means a) $A = \Delta_0(A) = \Delta_1(A)$,

b) $\exists x[A = \Delta_0(x)]$ or $\exists y[A = \Delta_1(y)]$,

c) $\forall x \forall u[\Gamma(x, A; u) \Rightarrow x = u]$ and $\forall y \forall v[\Gamma(A, y; v) \Rightarrow y = v]$.

That is, the three formulas a, b, c express provably equivalent properties of A, and this common property is that of being an *object*. It is usually understood that a capital letter used as a variable (free or bound) is restricted to refer only to objects.

Mono(f) means $\forall x \forall y[xf = yf \Rightarrow x = y]$.

Epi(f) means $\forall x \forall y[fx = fy \Rightarrow x = y]$.

Endo(f) means $\Delta_0(f) = \Delta_1(f)$.

Iso(f) means $\exists g[fg = \Delta_0(f)$ and $gf = \Delta_1(f)]$.

$A \cong B$ means $\exists f[A \xrightarrow{f} B$ and Iso$(f)]$.

A is a *retract* of B means $\exists f \exists g[A \xrightarrow{f} B$ and $fg = A]$.

G is a *generator* means $\forall f \forall g[\Delta_0(f) = \Delta_0(g)$ and $\Delta_1(f) = \Delta_1(g)$ and $f \neq g \Rightarrow \exists x[\Delta_0(x) = G$ and $\Delta_1(x) = \Delta_0(f)$ and $xf \neq xg]]$.

In a similar way a great number of the usual categorical notions can be expressed as formulas in the elementary theory of abstract categories;

for example, Prod $(A, B; P, p, q)$, meaning that P with projections p, q is a product of A with B, the notions of coproduct, terminal object, coterminal object, equalizer, coequalizer, meet (pullback), and comeet (pushout) are all elementary. However the notions of infinite limits and colimits, or of an object being "finitely generated" are not always elementary from the point of view of a given category, although they do become elementary if the category is viewed as an object in the category of categories, as explained below.

By a category we of course understand (intuitively) any structure which is an interpretation of the elementary theory of abstract categories, and by a functor we understand (intuitively) any triple consisting of two categories and a rule T which assigns, to each morphism x of the first category, a unique morphism xT of the second category in such a way that always

$$\text{if} \quad \Delta_i(x) = A\,, \quad \text{then} \quad \Delta_i'(xT) = AT \quad \text{for} \quad i = 0, 1\,,$$
$$\text{if} \quad \Gamma(x, y; u)\,, \quad \text{then} \quad \Gamma'(xT, yT; uT)\,.$$

Here "morphism" is the usual name for the "elements" of a category, the primes denote the interpretations of Δ_0, Δ_1, Γ in the second category, and calling T a "rule" is not supposed to have any connotation of effectiveness, etc.

With the evident definitions of Δ_0, Δ_1, Γ, the world of all functors becomes itself a category. Our purpose for the remainder of this article will then be to indicate certain axioms which hold for this intuitively-conceived category; actually there will be two theories, *a basic theory and a stronger theory.*

Both the basic theory and the stronger theory have the same notion of formula, which is essentially that of the elementary theory of abstract categories except that two individual constants ∂_0, ∂_1 are adjoined. These are needed in order to enable us to distinguish in a fixed way between a category and its dual, and they are intended to denote the two constant endofunctors of the ordinal number 2, considered as the category pictured below

$$\underset{0}{\cdot} \rightarrow \underset{1}{\cdot}\,.$$

Formally 2 is defined by (any one of) the equations

$$\Delta_i(\partial_j) = 2\,, \quad i, j = 0, 1\,.$$

Of course, now that we are in the category of categories, the things denoted by capitals will be called categories rather than objects, and we shall speak of functors rather than morphisms.

The axioms of the basic theory are those of the elementary theory of abstract categories plus several more axioms.

First we assume the existence of the category with exactly one morphism.

$$\exists 1 \, \forall A \, \exists ! x [A \xrightarrow{x} 1] .$$

A functor is called *constant* iff it factors through 1. We also find it a great notational convenience to assume the following "partial skeletal axiom":

$$\forall x [A \xrightarrow{x} A \quad \text{and} \quad \text{Iso}(x) \Rightarrow x = A] \quad \text{and} \quad A \cong B \Rightarrow A = B .$$

That is, if the identity is the only endofunctor of A which is an automorphism, then A is the only category in its isomorphism class. For example, 1 is the unique terminal category. We now state axioms characterizing 2:

∂_0 and ∂_1 are constant .
$\Gamma(\partial_i, \partial_j; \partial_j)$, $\; i, j = 0, 1$.
$\partial_0 \neq \partial_1$, $\;\partial_i \neq 2$, $\; i = 0, 1$.
$\forall x [2 \xrightarrow{x} 2 \Rightarrow x = \partial_0 \;\text{ or }\; x = \partial_1 \;\text{ or }\; x = 2]$.
2 is a generator.
If C is any generator, then 2 is a retract of C.

The intuitive validity of the last statement is easily seen with the help of the category E to be defined presently.

Proposition. If C is any generator with exactly three endofunctors, two of which are constant, and which is a retract of any other generator, then $C = 2$.

We remark that a simpler set of properties hoped by Freyd to characterize 2 [*Abelian Categories*, Harper and Row 1964] fails to do so since the following category also has exactly two objects and three endofunctors:

$$\cdot \underset{p}{\overset{i}{\rightleftarrows}} \cdot \circlearrowright a \qquad \begin{aligned} ip &= \text{identity}, \\ a &= pi . \end{aligned}$$

The symbols τ, $\bar{\partial}_i$ will be used to denote the unique functors making the following diagram commutative.

$$\begin{array}{ccc} 2 & \xrightarrow{\partial_i} & 2 \\ & \tau \searrow \quad \nearrow \bar{\partial}_i & \\ & 1 & \end{array} \qquad i = 0, 1 .$$

Basic is the following

Definition. $x \in A$ means $2 \xrightarrow{x} A$.

This will be read "x is a morphism in A". While this notation has a strong intuitive appeal, it should not be thought to have much formal connection with class elementhood in the usual sense; for example, if two categories have at least one morphism in common, they are equal.

Definition. If $x \in A$, then

$$A \models \Delta_i(x) = a \quad \text{means} \quad \partial_i x = a \qquad i = 0,1 .$$

These are read "a is the domain [respectively codomain] of x in A"; they clearly imply that $2 \xrightarrow{a} A$ also. Using this we can define $A \models \mathrm{Obj}\,(a)$ in the obvious way. We sometimes confuse an object $a \in A$ with the corresponding $1 \xrightarrow{\bar{a}} A$ such that $a = \tau\bar{a}$. Note that it is provable that

$$\partial_0 \xrightarrow{2} \partial_1 \quad \text{"in 2"} .$$

Axiom of Finite Roots. *There is a coterminal category* 0. *Any two categories have a product and a coproduct. Any two functors with a common domain category and a common codomain category have an equalizer and a coequalizer.*

It is well known that meets and comeets, etc., in particular inverse images and intersections can then be proved to exist. We also assume at this point the following axiom.

If $A \xrightarrow{i} A + B \xleftarrow{j} B$ is a coproduct diagram, then

$$x \in A + B \Rightarrow \exists\, y\, [x = y\, i] \quad \text{or} \quad \exists\, z\, [x = z\, j] .$$

Incidentally, if

$$1 \underset{\bar{\partial}_1}{\overset{\bar{\partial}_0}{\rightrightarrows}} 2 \to N$$

is a coequalizer diagram, then we call N the additive monoid of non-negative integers. This shows that the basic theory needs no explicit "axiom of infinity".

Another consequence of the axiom of finite roots is that the colimit of the following diagram exists:

$$\begin{array}{ccc} 1 & \xrightarrow{\bar{\partial}_0} & 2 \\ & \bar{\partial}_1 \searrow \nearrow \bar{\partial}_0 & \\ 1 & \xrightarrow[\bar{\partial}_1]{} & 2 \end{array}$$

Denote this colimit by E and the two injections $2 \to E$ by φ and ψ.

Axiom. *The category E has exactly four morphisms, namely*

$$\varphi,\ \psi,\ \partial_0 \psi = \partial_0 \varphi ,\quad \partial_1 \psi = \partial_1 \varphi .$$

This axiom may well be provable from the others; at any rate it allows us to picture the "inside" of E as follows:

$$\dot{0} \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \dot{1}$$

Although we have characterized 2, we still have not assumed enough about it, for all axioms stated so far are valid in the category of directed graphs (in particular our remark about N is really only sensible in view of axioms still to be stated). We need another

Definition. 3, α, β will always mean the unique category and functors in the following *comeet* (pushout) diagram:

$$\begin{array}{ccc} 1 & \xrightarrow{\bar{\partial}_1} & 2 \\ \bar{\partial}_0 \downarrow & & \downarrow \alpha \\ 2 & \xrightarrow[\beta]{} & 3 \end{array}$$

Axiom. 3 *has exactly one morphism* γ *besides the five implied by the definition (which are distinct); it satisfies*

$$\partial_0 \gamma = \partial_0 \alpha, \quad \partial_1 \gamma = \partial_1 \beta .$$

Thus 3 may be pictured internally:

$$\begin{array}{ccc} \overset{0}{\bullet} & \xrightarrow{\alpha} & \overset{1}{\bullet} \\ & \gamma \searrow & \downarrow \beta \\ & & \underset{2}{\bullet} \end{array}$$

Using 3, α, β, γ we can make the all-important

Definition. If $f \in A$, $g \in A$, $h \in A$, then

$$A \models \Gamma(f, g; h) \quad \text{means} \quad \exists t[\alpha t = f,\ \beta t = g,\ \gamma t = h].$$

This is read "h is the composition f followed by g in A". For example, $3 \models \Gamma(\alpha, \beta; \gamma)$ can be proved.

We can also now formally prove that every f in our world is a functor according to our earlier intuitive definition. Of course such a t as above must satisfy $3 \xrightarrow{t} A$; hence the letter t for "triangle".

We have defined $A \models \Delta_0$, $A \models \Delta_1$, $A \models \Gamma$. Now given any formula Φ of the elementary theory of abstract categories, we can make in Φ the following substitutions

$A \models \Delta_i$ for Δ_i $i = 0,1$
$A \models \Gamma$ for Γ
$\forall x[x \in A \Rightarrow\ ]$ for $\forall x[\ ]$
$\exists x[x \in A$ and $]$ for $\exists x[\ ]$

and thus obtain a new formula $A \models \Phi$ in the basic theory which has one more free variable (namely A) than Φ has, and which expresses intuitively the statement that Φ is true in A (of the morphisms in A denoted by the free variables if there are any). If Φ is a sentence of the elementary theory of abstract categories and if $A \models \Phi$ holds, we say that A is a *model* for Φ or that Φ is *true in* A. Similar remarks hold for formulas of the basic theory if ∂_0, ∂_1 are thought of as variables in the modified formula. We have not yet described all of the basic theory; however its axioms will only be finite in number, so the conjunction of all of them (more precisely of their universally quantified closures) will be a single sentence. Thus

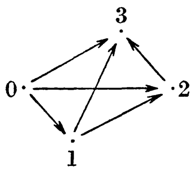
"there exists a model A of the basic theory having property Ψ"
(where Ψ is any other formula of the basic theory) is also a formula of the basic theory, which will be a sentence if Ψ has only one free variable A. Of course such a formula could not be proved in the basic theory; in fact, the passage from the basic theory to the stronger theory will involve just the addition of two formulas of the above type to the list of axioms.

At present, we have not yet assumed enough to insure that we are not talking about a category of objects with a non-associative partial multiplication. We now remedy this.

Definition. 4 is the category in the comeet diagram.

$$\begin{array}{rcc} & \mathbf{1} \xrightarrow{\bar{\partial}_0} \mathbf{2} & \\ \bar{\partial}_1 \gamma \downarrow & & \downarrow \\ & \mathbf{3} \rightarrow \mathbf{4} & \end{array}$$

Axiom. 4 *has exactly ten morphisms, satisfying the evident equations so that* 4 *may be pictured*:



Evidently the above picture is closely related to the diagram used to describe the associativity axiom in our discussion of the elementary theory of abstract categories. (Let

$$0 \xrightarrow{x} 1 \xrightarrow{y} 2 \xrightarrow{z} 3\,.$$

Then f, g are represented by the same arrow in the above picture.) In fact

Theorem Schema. *If Φ is any theorem of the elementary theory of abstract categories, then*

$$\forall A\,[A \models \Phi]$$

is a theorem of the basic theory of the category of all categories.

Thus every object in a world described by the basic theory is at least a category. The remaining five axioms of the basic theory are intended to help insure that the objects are no more than categories (i.e. have no further structure) by insuring that there are many functors, and also to help make sure that there are enough categories. The additional two axioms of the stronger theory will have also the latter aim.

One of the most important constructions of category theory is the formation of functor categories, and of course our intuition tells us that whenever two categories exist in our world, then so does the corresponding category of all natural transformations between the functors from the first category to the second. Thus

Axiom. Given two categories A, B, there is a category B^A and a functor (called evaluation)

$$A \times B^A \xrightarrow{e} B$$

such that for any C and for any $A \times C \xrightarrow{f} B$ there is exactly one $C \xrightarrow{h} B^A$ such that

$$f = (A \times h)\,e. \qquad \begin{array}{ccc} A \times C & & \\ A\downarrow \times \downarrow h & \searrow f & \\ A \times B^A & \xrightarrow{e} & B \end{array}$$

[The above statement of the exponentiation axiom is not quite precise since $A \times h$ is not meaningful until projections have been chosen for the two products. The intention is that the axiom as stated holds for any choice of projections (and of the product categories themselves) with the correct universal properties; this can easily be written out directly in the elementary language in about a page and a half.]

The exponentiation axiom implies that products distribute over sums, that the usual laws of exponents hold, and that for any three categories A, B, C there is a single "composition functor"

$$B^A \times C^B \xrightarrow{\circ} C^A.$$

To deduce the usual internal description, let $C = 1$ in the exponentiation axiom; from the fact that $A \times 1 \cong A$, it then follows immediately that the objects in B^A correspond exactly to the functors from A to B. To see that the morphisms in B^A correspond exactly to the "natural transformations" between such functors, set $C = 2$ in the axiom, use commutativity of products and the fact that a functor $A \to B^2$ must have commutative squares in B as values since 2×2 has an easily-deduced internal picture, and verify that the equations which must be satisfied at each stage of the just-indicated transformations imply that all these individual squares fit together as they should*. If $2 \xrightarrow{\varphi} B^A$ is a morphism in B^A, then the functors $A \to B$ corresponding to its domain and co-domain in B^A are explicitly

$$A \cong A \times 1 \xrightarrow{A \times \bar{\partial}_i} A \times 2 \xrightarrow{\bar{\varphi}} B \qquad i = 0,1$$

where $\bar{\varphi} = (A \times \varphi)\, e$, the e being of course the evaluation functor. If $A \xrightarrow{f} B$, we denote by 1_f the object of B^A corresponding to it. In particular 1_A is a distinguished object $2 \to A^A$ in A^A, and $1_{fg} = 1_f \circ 1_g$.

Definition. The category A is said to be *discrete* (or to be a *set*) iff A^τ is an isomorphism $A^2 \xrightarrow{\sim} A^1$. That is, every morphism in a set is an object.

Axiom. *For any category A there is a discrete category A_c with a functor $A \to A_c$ such that for any functor $A \to B$ from A to a discrete category there is exactly one functor making this diagram commute*

$$\begin{array}{ccc} A & \to & A_C \\ & \searrow & \downarrow \\ & & B \end{array} .$$

This A_c is called the *set of components* of A.

Axiom. Dualize the proceeding axiom. Thus every category A has a maximal discrete subcategory $|A|$, called for obvious reasons the *set of objects* of A. The "absolute value" notation for the set of objects will be used consistently. By the *set of morphisms* of A we understand the discrete category $|A^2|$, since A^2 is of course a category whose objects correspond to morphisms in A. In particular

$$|N^2|$$

is called the *set of nonnegative integers*, where N is the monoid of non-negative integers as previously defined.

We also state now the

Axiom of Choice.

$$A \xrightarrow{f} B,\ 0 \not\cong A\,,\ B \text{ discrete} \Rightarrow \exists g[fgf = f]\,.$$

Now it follows easily from the definition that if B is discrete, so is B^C for any C. Also, since the usual formal proof that adjoints preserve

* B^2 is not to be confused with $B \times B$; $\bar{\partial}_0$, $\bar{\partial}_1$ induce a functor $B^2 \to B \times B$ which is in general faithful but not full.

limits holds equally well for metacategories and metafunctors (i.e. "subcategories" of the universe defined by formulas but which, like the full metacategory of all sets, cannot necessarily be represented by an actual category in the universe,) it follows that sets are closed under the formation of finite roots. In fact,

Metatheorem. *Define a relative interpretation of the elementary theory of abstract categories into the basic theory by relativizing all quantifiers to functors between discrete categories. Then in the induced theory, all theorems of the elementary theory of the category of sets* [LAWVERE, Proc. Nat. Ac. Sc. USA Dec. 1964] *are provable.*

Thus one could, by referring only to discrete categories, develop on the basis of the axioms we have so far assumed such subjects as number theory, calculus, linear operators in Hilbert space, etc. (such a program, of course, would not make the most efficient use of the functorial method.) In such a development, as well as in our work here, it is convenient to use the following metatheorem, which, because it is provable in the elementary theory of the category of sets, is by the above also available in the basic theory of the category of categories.

Predicative Subset Schema. *Suppose that Φ is any formula of the elementary theory of abstract categories whose free variables are $A_0, \ldots, A_{m-1}, x_0, \ldots, x_{n-1}$, and in which all bound variables are restricted to range only over morphisms whose domains and codomains are among the A_j, $j < m$.* Let *σ_0, σ_1 be any mappings from n to m, and think of x_i as ranging over morphisms $A_{i\sigma_0} \to A_{i\sigma_1}$ for $i < n$. Then (it is provable in the elementary theory of the category of sets that) there exists a subset y_Φ of*

$$\prod_{i<n} A_{i\sigma_1}^{A_{i\sigma_0}}$$

such that the members of y_Φ are exactly those elements $\langle x_0, \ldots, x_{n-1}\rangle$ of the above product for which

$$\Phi(A_0, \ldots, A_{m-1}, x_0, \ldots, x_{n-1})$$

is true.

In the above assertion the definitions of subset, member, and element are those given in the above-cited article. Note that as a special case one could have $A_{i\sigma_0} = 1$ and $A_{i\sigma_1} = A$ for all $i < n$, so that in particular all predicatively definable relations on a given set A exist.

The last axiom of the basic theory is intended to express that the full (and finite) metacategory determined by 1, 2, 3 is *adequate* in the universe in the sense of ISBELL [Rozprawy Matematyczne XXXVI]. Of course, no elementary axiom could really express this, but at least we can express adequacy relative to the metacategory of sets as we have defined that. Essentially the axiom states that when given a function from the set of morphisms of a category A into the set of morphisms of a category B, if it satisfies the intuitive definition of "functor", then we can find in the universe the corresponding actual functor from A itself to B itself.

Axiom. *If* $|A^2| \xrightarrow{\bar{f}} |B^2|$ *and*

$$\forall t[3 \xrightarrow{t} A \Rightarrow \exists!\, u[3 \xrightarrow{u} B \text{ and } |u^2| = |t^2|\bar{f}]]$$

then

$$\exists!\, f[A \xrightarrow{f} B \quad \text{and} \quad \bar{f} = |f^2|].$$

Having presented the axioms for the basic theory of the category of categories, we now ask what can be done with them. Besides the possibility of developing analysis which was previously alluded to, one can also define easily the full metacategories of ordered sets, groups, or algebraic theories [LAWVERE, Proc. Nat. Ac. Sc. USA, Nov. 1963] and study these to a considerable extent; however comfortably complete *categories* (i.e. objects in the universe) corresponding to these cannot be shown to exist without adding the two axioms of the stronger theory which will be discussed presently. For any category A, the category of semisimplicial objects from A can also be shown to exist in the basic theory, although a much less messy proof can be given in the stronger theory. The general theories of triplable categories, of fibered categories, and of closed categories (when the latter is phrased so as not to refer to the category of sets) can all be developed quite nicely within the basic theory, as can many other things. Thus before we state the stronger axioms, we will discuss some principles which can be proved using only the basic theory.

First we point out that of the several definitions of "adjoint functors", all except the one involving hom-functors can be easily stated in the basic theory. The following general adjoint functor theorem can then also be proved in the basic theory.

Theorem. *A functor* $A \xrightarrow{f} B$ *has an adjoint iff*

i) *f preserves all (inverse) limits which exist in A.*

ii) *For every object $b \in B$, the category (b, f) has a final subcategory C_b which is among those over which A has (inverse) limits.*

Here (b, f) is a special case of an operation defined below, and to say that $C_b \to (b, f)$ is *final* is meant in the following sense:

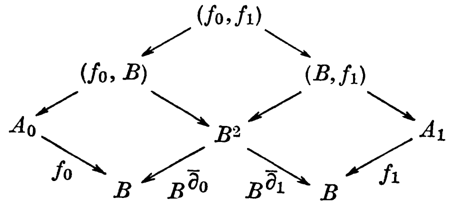
> $C \xrightarrow{u} C'$ is *final* iff for every g such that $\Delta_0(g) = C'$, if $\varprojlim (ug)$ exists, so does $\varprojlim (g)$ and
>
> $$\varprojlim (g) \cong \varprojlim (ug)$$
>
> in $\Delta_1(g)$.

All other adjoint functor theorems are refinements of the above intended to make condition (ii) easier to verify in special cases. Usually one employs a notion of "small" category (which will be available in our stronger theory) and assumes that A is complete (has limits over small categories); then one need only say in condition (ii) that C_b is small and final in (b, f) for each b. Sometimes it is useful also to assume that the hom-sets (defined below) of A and B are small. The existence of such C_b's in this context is easily seen to be equivalent to Freyd's Solution-Set Condition [*Abelian Categories*].

The following operation is very convenient, and easily seen to exist in the basic theory. Given two functors

$$A_i \xrightarrow{f_i} B \qquad i = 0,1$$

with a common co-domain, define the category (f_0, f_1) so that all three squares below are meets (pullbacks).



Note that (except for the canonical isomorphism $B^1 \cong B$) B^{∂_i} represent the domain and codomain functors on B. There is a forgetful functor

$$(f_0, f_1) \to A_0 \times A_1$$

each object of (f_0, f_1) having the additional structure involving a morphism in B.

We consider some special cases of the (,) notation. If $A_0 = f_0 = B$, and if $A_1 = 1$, so that $f_1 = b$ is an object in B, then

$$(B, b)$$

is nothing but the category of "objects over b" as used for example by Beck in his triple cohomology [this volume]. If both A_0, A_1 are 1, then one can show that

$$(b_0, b_1)$$

is a discrete category, called the set of B-morphisms from b_0 to b_1, or simply a *hom-set*. (This does not mean, by the way, that a hom-functor exists for B. For one thing, we cannot show in the basic theory that a category of sets exists, and in any case no single category of sets could serve as the recipient of hom-functors for all categories B.) The third special case which we consider is that where $B = f_1 = A_1$ is a monoid (category with one object) with $A_0 = 1$ and $f_0 = e =$ the unique functor $1 \to B$. Then

$$(e, B)$$

is a category which contains the divisibility information about B. If the monoid B has cancellation, then (e, B) is a preorder (category in which every hom-set is 0 or 1) and in particular we define

$$\omega = (0, N)$$

the *well-ordered set of natural numbers.*

And of course still another case of the (,) notation was used in the statement of the adjoint functor theorem.

The following theorem has somewhat the same sort of use in constructing categories that the adequacy axiom has in constructing functors. Notice that the hypothesis of the theorem describes essentially a set equipped with a partial multiplication table which satisfies the axioms of the elementary theory of abstract categories.

Theorem. *Let* A_1, A_2, A_3, A_4 *be given discrete categories, and let* $A_2 \xrightarrow{d_i} A_2$, $i = 0{,}1$ *and* $A_3 \xrightarrow{c} A_2$ *be given functors, satisfying the following conditions:*

$$d_i\, d_j = d_i \qquad i, j = 0{,}1\,.$$

There is given $A_1 \to A_2$ *which is the equalizer of* d_0 *with* d_1. A_3 *is the meet of* d_0 *with* d_1, *with structural functors* a_i:

$$\begin{array}{ccc} A_3 & \xrightarrow{a_0} & A_2 \\ {\scriptstyle a_1}\downarrow & & \downarrow{\scriptstyle d_1} \\ A_2 & \xrightarrow[d_0]{} & A_2 \end{array}$$

A_4 *is the meet of* a_0 *with* a_1, *with structural functors* b_i:

$$\begin{array}{ccc} A_4 & \xrightarrow{b_0} & A_3 \\ {\scriptstyle b_1}\downarrow & & \downarrow{\scriptstyle a_1} \\ A_3 & \xrightarrow[a_0]{} & A_2 \end{array}$$

The functor $A_3 \xrightarrow{c} A_2$ *satisfies the "book-keeping" axioms:*

$$c\, d_i = a_i\, d_i \qquad i = 0{,}1$$

as well as the identity axioms

$$e_i\, c = A_2 \qquad i = 0{,}1$$

and the associativity axiom

$$f_0\, c = f_1\, c$$

where $A_4 \xrightarrow{f_i} A_3$ *and* $A_2 \xrightarrow{e_i} A_3$ *are constructed in the evident fashion.*

Conclusion. *There is a category A together with isomorphisms*

$$A_i = |A^i| \qquad i = 1, 2, 3, 4$$

so that (in addition to obvious compatibility conditions)

$$\begin{aligned} d_i &= |A^{\partial_i}| \qquad i = 0,1 \\ c &= |A^{\gamma}|. \end{aligned}$$

Any two such categories are canonically isomorphic.

In the proof of the above theorem, A is constructed as a quotient of the free category $A_2 \times 2$, the two functors along which the coequalizer is taken being constructed with the help of the given d_0, d_1, c.

Corollary. *Every category A has a dual A^*.*

For let $A_2 = |A^2|$, $d_0 = |A^{\partial_1}|$, $d_1 = |A^{\partial_0}|$, etc.

Combining the above theorem with the Predicative Subset Schema for sets and with the adequacy axiom, one can derive

Predicative Functor-Construction Schema. *Let $\Phi(a, b)$ be a formula (possibly with parameters) such that all bound variables are suitably restricted, and suppose that*

$$\begin{aligned} &\forall a \in A\ \exists!\, b \in B[\Phi(a, b)] \\ &\Phi(a, b) \Rightarrow \Phi(\partial_i a, \partial_i b) \qquad i = 0,1 \\ &\Phi(a_i, b_i) \quad i = 0, 1, 2 \quad \text{and} \quad A \models \Gamma(a_0, a_1; a_2) \\ &\qquad\qquad\qquad\qquad \Rightarrow B \models \Gamma(b_0, b_1; b_2). \end{aligned}$$

Then

$$\exists!\, f[A \xrightarrow{f} B \quad \text{and} \quad \forall a\, \forall b [a f = b \Longleftrightarrow \Phi(a, b)]].$$

Often in applying this schema (for example in the proof of the general adjoint functor theorem stated earlier), the given formula Φ does not quite have the absolute uniqueness property stated in our above hypothesis, but has it only up to isomorphisms in B which are themselves unique with respect to some other parameters (projections, etc.) in the formula. However, with the help of the axiom of choice a functor f as desired can still be constructed; of course it will itself only be unique up to natural equivalence.

Another corollary of our theorem on construction of categories with given set of morphisms and given multiplication table is the following

Theorem. *Let B_0, $B_1, \ldots, B_{n-1}$ be a finite number of categories, assumed distinct. Then there exists a category with n objects which is "isomorphic" to the full metacategory of all functors between the B_i's.*

Here the formal significance of the word "isomorphic" can be guessed from the construction which proves the theorem, this construction beginning by setting

$$A_2 = \sum_{i,j<n} | B_j^{B_i} |$$

Definition. *For any category A, let $\{A\}$ denote the category with four objects obtained by applying the above theorem to the list of four categories*

$$1, 2, 3, A .$$

(We do not bother to give a definition of $\{A\}$ in the three exceptional cases where A is 1 or 2 or 3.)

The reason for this "singleton" notation is that, intuitively, if C is a category of categories (i.e. a model for the basic theory), then A is ("isomorphic" to) an object in C iff $\{A\}$ is in a smooth way a subcategory of C. This will be made more precise before we state the axioms of the stronger theory. It makes sense to say that a category C is a model for the basic theory since, because the basic theory is finitely axiomatized, the conjunction of all its axioms is a single sentence Φ of the basic theory. (Strictly, it makes sense only relative to a given pair $2 \rightrightarrows C$ of morphisms in C which can play the roles of ∂_0, ∂_1 in C). Now of course (assuming consistency) we cannot prove in the basic theory that there exist models for the basic theory. However, it is useful to know that we can prove in the basic theory that its models are (essentially) just as numerous as models for the elementary theory of the category of sets.

Theorem. *Suppose C is any model for the basic theory. Then the full subcategory determined by the discrete objects in C is a model for the elementary theory of the category of sets. (Such a subcategory exists as the equalizer of the identity functor C with endofunctor of C corresponding to the notion $|\ \ |$). On the other hand, if Q is any model for the elementary theory of the category of sets (which also has a finite number of axioms) then there is a full subcategory of the functor category*

$$Q^{\{4\}*}$$

which is also constructible as a finite left root and which is (except for a minor adjustment to account for the partial skeletal axiom) a model of the basic theory.

The subcategory of the functor category in question is that determined by those contravariant functors from $\{4\}$ to Q which take comeets (of these finite ordinals) into meets (of "sets"). Notice that the $a_0, a_1, b_0, b_1, c, d_0, d_1, e_0, e_1, f_0, f_1$ occurring in our theorem of the construction of categories from multiplication tables have exactly the form of the category $\{4\}^*$, with A_i corresponding to i. Of course, by adequacy any category of categories is represented fully by a category of diagrams of sets which have only three vertices, but 4 is needed here since the category of such diagrams would contain many objects with *non-associative* partial multiplications.

We now prepare to state the axioms of the stronger theory with some definitions.

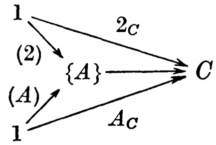
Definition. *A $(\in)$ C means that C is a model of the basic theory and that there is a full, faithful, root-preserving functor*

$$\{A\} \to C$$

which also preserves the morphisms with the names ∂_0, ∂_1.

Proposition. *A functor $\{A\} \to C$ as described in the above definition is unique if it exists (up to natural isomorphism).*

Notice that if A $(\in)$ C, then there are two objects A_C, 2_C in C with the correct significance:



and that one has canonically

$$|A^2| \cong (2_C, A_C).$$

Thus, if A $(\in)$ C, B $(\in)$ C, and if $2 \xrightarrow{x} C$ is such that

$$C \models [\Delta_0(x) = A_C \quad \text{and} \quad \Delta_1(x) = B_C]$$

then there is an induced functor "$(2_C, x)$"

$$|A^2| \to |B^2|.$$

Definition. If $A \xrightarrow{f} B$, A $(\in)$ C, B $(\in)$ C, then

$$f \, (\in) \, C$$

shall mean that some $x \in C$ induces $|f^2|$.

Definition. *C is full in the universe means that C is a model of the basic theory and that whenever A $(\in)$ C, B $(\in)$ C, and*

$$A \xrightarrow{f} B$$

one has

$$f \, (\in) \, C.$$

Definition. *A is C-complete means that C is a model of the basic theory and for every functor f, if f $(\in)$ C then A^f has a co-adjoint. Dually, A is C-co-complete means that every such A^f has an adjoint.*

Actually only functors f with co-domain 1 have to do with limits as such, since they induce the diagonal functors

$$A \to A^D \quad D = \Delta_0(f).$$

However, it is well known that if A has enough co-limits, then lots of induced functors

$$A^{D'} \xrightarrow{A^f} A^D, \quad \text{where} \quad D \xrightarrow{f} D'$$

have adjoints, and since these arise very often (e.g. in algebra and sheaf theory) it seems more to the point to incorporate these directly into the definition of co-completeness.

Axiom. *For any A, and for any model C of the basic theory, there exists a smallest category*

$$C[A]$$

which is a model of the basic theory, which is C-complete and full in the universe, and for which

$$A \ (\in)\ C.$$

Here "smallest" means up to equivalence of categories. Actually the above axiom is not very strong since it does not give us the "inaccessible" category which we need (in fact we still cannot prove that there exist any models of the basic theory). Thus

Axiom. *There exists a category C_0 which is full in the universe (in particular is a model of the basic theory) and such that*

$$C_0 \textit{ is } C_0\textit{-complete.}$$

Further, any category satisfying these conditions is equivalent to C_0.

The last clause thus embodies the idea that only *one* inaccessible is needed for most mathematics; our world thus stops far short of the second Grothendieck universe if we assume the above axiom. Why not much category theory is lost thereby will be explained below. We could have of course assumed much stronger axioms. For example, by analogy with the work of the set-theorists BERNAYS and LEVY, we could alternately have assumed the following infinite set of axioms.

Strong Reflection Principle: Let Φ be any formula, with free variables $v_0, v_1, \ldots, v_{n-1}$. Then
$\Phi(v_0, v_1, \ldots, v_{n-1}) \Rightarrow \exists C$ [C is C-complete and full in the universe and $v_0, \ldots, v_{n-1} \ (\in)\ C$ and $C \models \Phi(\bar{v}_0, \ldots, \bar{v}_{n-1})$].
(Here $\bar{v}_i$ denotes the morphism in C corresponding to the functor v_i.)

However, we remain in this article with the finite list of axioms which we have presented, and call it the stronger theory (although as we have just pointed out there are much stronger ones still).

Definition. The full subcategory of C_0 determined by its discrete objects is denoted by S and called the *category of sets* (more precisely the category of small sets). A category A is *small* iff A $(\in)$ C_0. A category is *complete* iff it is C_0-complete (i.e. has small limits). We write

$$\widetilde{C} = C_0[S].$$

Thus $\widetilde{C}$ is the smallest full, complete category of categories which contains the category of small sets as an object; $\widetilde{C}$ is itself an object in our world.

Actually most mathematics, including most category theory, can be done if we assume only the existence of S and $\widetilde{C}$ and the basic theory, provided we understand that structures are always to be small (i.e., modeled in S), at least whenever we collect structures into categories. For example, semantics functors for categories of small theories [Lawvere, Proc. Nat. Ac. Sc. USA, Nov. 1963 (Algebraic Theories) and Logic Colloquium Leicester 1965 (Elementary Theories)] all take their values in a part of $\widetilde{C}$, and all the usual examples of large fibered categories also involve only a "small" part of $\widetilde{C}$.

Theorem. *If A has small hom-sets then there is a hom-functor*

$$A^* \times A \to S.$$

We leave to the reader to make precise within our language what it means to be a hom-functor. Applying the exponentiation axiom and the usual argument for Yoneda's lemma, we obtain the usual.

Corollary. *If A has small hom-sets, then there is a full and faithful functor*

$$A \to S^{A^*}$$

which preserves any (inverse!) limits which may exist in A.

The above representation is of course the starting point of most investigations into the structure of categories (see for example the work of Freyd, Mitchell, Lawvere, Isbell, Linton, etc.), the aim being in general to cut down on the size of A^* and to say more about the image of the representation. This is also the basic method used in proving the following, which in order to make contact with previous work in foundations, we have phrased in the language of set theory, although it could also be phrased in the stronger theory itself (if the uniqueness of $\mathfrak{C}_0$ is dropped).

Metatheorem. Let θ_2 be the third stongly inaccessible ordinal (where the first is $\theta_0 = \omega$) and let $\mathfrak{C}$ be category whose morphisms are all functors (defined in the obvious set-theoretical way as triples) whose domain and codomain categories have their underling sets of rank less than θ_2. Then $\mathfrak{C}$ is a model for the stronger theory (in particular models for the stronger theory cannot contain an element anything like θ_2 itself). Conversely, given any category $\mathfrak{M}$ which is a model for the stronger theory, and all of whose hom-sets have cardinality less than θ_2, there is a functor

$$\mathfrak{M} \to \mathfrak{C}$$

which is an equivalence of categories if $\mathfrak{M}$ has products of size α for any $\alpha < \theta_2$. Actually, not θ_2 but $\theta_1 + \theta_1$ gives the smallest "natural" model for the stronger theory; this smallest natural model thus has cardinality $\beth_{\theta_1}(\theta_1)$ which is the number reached by starting at the first inaccessible beyond ω (namely θ_1) and iterating the power set operation θ_1 times.

We conclude by posing what seems to be a basic open problem in the foundations of category theory.

Problem. *Find a useful characterization of those complete categories A with small hom-sets such that every functor*

$$A \to S$$

which preserves (inverse) limits has an adjoint.

The aim of the problem is clearly to understand when one can ignore condition (ii) in the General Adjoint Functor Theorem. Such categories A do exist, by FREYD's Special Adjoint Functor Theorem [*Abelian Categories*] where (ii) is replaced by the assumption that A have a cogenerator and that A be well-powered. ISBELL's notion co-adequacy seems to be relevant to the problem. More particulary, one can ask which A's among some known class, say that of algebraic categories, have the property in question.

Footnote added 22 Oct. 1965: Professor ISBELL has since shown the author an example of functor (which can be constructed in our theory) from small groups to small sets which is left continuous but not representable by a small group.

Forschungsinstitut für Mathematik
Eidgenössische Technische Hochschule
Zürich