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# ALGEBRAIC THEORIES, ALGEBRAIC CATEGORIES, AND ALGEBRAIC FUNCTORS

F. WILLIAM LAWVERE

*Reed College, Portland, Oregon, U.S.A.*

The theory of categories and functors (Eilenberg-MacLane [45], Freyd [60], Kan [58], MacLane [50]) extends to the various fields of mathematics a methodological injunction which may be briefly phrased: "Look to the maps." That such a program can lead to interesting results in the field of model theory is partly borne out in the case of the study of "equational classes of algebras" by the results discussed below. There are at least four distinct levels in general algebra where maps are to be seen: (1) Homomorphisms between algebraic structures of a given equational type. This suggests the notion of "algebraic category." (2) "Equational" interpretations between algebraic theories; for example, the obvious interpretation of the theory of Lie algebras into that of associative algebras. (3) "Algebraic" functors between algebraic categories induced by the interpretations just mentioned, as well as adjoints (Kan [58]) to these. The notion of reduct provides a simple, but by no means trivial, class of examples of algebraic functors; previously known examples of adjoints to reducts include the constructions of tensor algebras and semi-group rings as well as the embedding of a distributive lattice in a Boolean algebra. (4) The semantical assignment of algebraic categories to algebraic theories. This assignment is a functor which, we discover, also has an adjoint when properly construed. The adjoint provides us with a useful tool for giving a new intrinsic characterization of algebraic categories and also for partially analyzing categories of models of more general theories. The results to be discussed appeared in an earlier form in the author's Columbia University doctoral dissertation (Lawvere [63]) written under the direction of Professor Eilenberg. For more recent related material see Isbell [a], Lawvere [a], and unpublished work of F. E. J. Linton.

By an *algebraic theory* we mean a small category $\boldsymbol{A}$ whose objects are the natural numbers 0, 1, 2, . . . and in which each object $n$ is the categorical direct product of the object 1 with itself $n$ times. By an *n-ary* *operation* of $\mathbf{A}$ is meant any map $n \to 1$ in $\mathbf{A}$. Since $n$ is a product, the projections $\pi_i^{(n)}: n \to 1$, $i = 0, 1, \ldots, n-1$ are always $n$-ary operations for each $n$ in any algebraic theory, but in general there will be more. The maps $n \to m$ in an algebraic theory $\mathbf{A}$ are in one-to-one correspondence with the $m$-tuples of $n$-ary operations of $\mathbf{A}$. Any "presentation" of a concept of algebraic structure (e.g., groups, modules over a given ring, Jordan algebras, lattices, etc.), which involves a set of symbols denoting finitary operations together with a set of equations (= identities) relating composite operations, determines an algebraic theory, and conversely every algebraic theory has such presentations. By a mapping between algebraic theories we will understand a functor that preserves products and takes 1 into 1. Algebraic theories and the mappings between them thus form a category $\mathscr{T}$.

Each algebraic theory $\mathbf{A}$ determines a large category $\mathscr{S}^{(\mathbf{A})}$ whose class of objects is just the equational class (variety) of all algebras of type $\mathbf{A}$, and whose maps are all (into) homomorphisms between these. An algebra of type $\mathbf{A}$ can be viewed as a product preserving functor $\mathbf{A} \to \mathscr{S}$ from $\mathbf{A}$ to the category of sets; a homomorphism of algebras is then just a natural transformation between such functors. If $\mathbf{A}$ is the algebraic theory whose *only* $n$-ary operations are projections (i.e., $\mathbf{A}$ is equivalent to the dual of the category of finite sets) then the category of algebras $\mathscr{S}^{(\mathbf{A})}$ is just the category $\mathscr{S}$ of sets. Every map $f: \mathbf{A} \to \mathbf{B}$ of algebraic theories determines in an obvious way a functor $\mathscr{S}^{(f)}: \mathscr{S}^{(\mathbf{B})} \to \mathscr{S}^{(\mathbf{A})}$ which preserves underlying sets, i.e., for which $U_{\mathbf{B}} = \mathscr{S}^{(f)} U_{\mathbf{A}}$, where $U_{\mathbf{A}}: \mathscr{S}^{(\mathbf{A})} \to \mathscr{S}$, $U_{\mathbf{B}}: \mathscr{S}^{(\mathbf{B})} \to \mathscr{S}$ are the underlying set functors (notice the order in which we write composition.) We call any functor of the form $\mathscr{S}^{(f)}$ an *algebraic functor*, and we call any category of the form $\mathscr{S}^{(\mathbf{A})}$ an *algebraic category*. Any algebraic theory $\mathbf{A}$ is equivalent to the dual of the full category of finitely generated free algebras in its associated algebraic category.

Theorem. *Every algebraic functor has an adjoint.*

For example, the category of rings (with unit) and that of monoids are algebraic, and the functor which assigns to each ring the monoid consisting of the same elements under multiplication alone is an algebraic functor. The adjoint to this functor is the well-known construction of the monoid ring. Other instances of this theorem include abelianization of groups, enveloping algebras of Lie algebras, and a generalized quaternion construction. The proof of the above theorem may be conveniently divided into two steps. Given $f: \mathbf{A} \to \mathbf{B}$ in $\mathscr{T}$, call $\mathscr{S}^{\mathbf{A}}$, the category of *all* functors $\mathbf{A} \to \mathscr{S}$ and natural transformations, the category of *$\mathbf{A}$-prealgebras;* similarly $\mathscr{S}^{\mathbf{B}}$ is the category of $\mathbf{B}$-prealgebras. The induced functor $\mathscr{S}^{\mathbf{B}} \to \mathscr{S}^{\mathbf{A}}$ has an adjoint which may be calculated by a direct limit procedure (see Lawvere [63, p. 39].) Secondly, the inclusion $\mathscr{S}^{(\mathbf{B})} \to \mathscr{S}^{\mathbf{B}}$ of algebras into prealgebras has an adjoint which may be calculated by considering, for a given prealgebra $\mathbf{B} \to \mathscr{S}$, the free algebra over the one-dimensional part reduced modulo the relations implied by the structure of the zero- and higher-dimensional parts. Note that for $f: \mathbf{A} \to \mathbf{B}$ in $\mathscr{T}$, the natural transformation from the identity functor on $\mathscr{S}^{(\mathbf{A})}$ to $F\mathscr{S}^{(f)}$, where $F$ is the adjoint of $\mathscr{S}^{(f)}$, need not be a monomorphism (for example, not every Jordan algebra is special—on the other hand every distributive lattice can be embedded in a Boolean ring); an important problem remaining is an intrinsic characterization of those $f$ in $\mathscr{T}$ for which the natural transformation in question is in fact a monomorphism.

If we write $\mathbf{A}\mathfrak{S} = U_{\mathbf{A}}$, $f\mathfrak{S} = \mathscr{S}^{(f)}$, we obtain a functor $\mathfrak{S}$ which we call *algebraic semantics;* the domain of $\mathfrak{S}$ is the dual $\mathscr{T}^*$ of the category of algebraic theories, and we take as its codomain the category $\mathscr{K}$ whose objects are functors $U: \mathscr{X} \to \mathscr{S}$ with arbitrary domain category and with the category of sets as codomain, subject only to the restriction that for each natural number $n$, the class of all natural transformations $U^n \to U$ is small, where $U^n$ assigns to each $X$ the $n$th Cartesian power of $XU$; a map $T: U \to U'$ in $\mathscr{K}$ is to be any functor $T: \mathscr{X} \to \mathscr{X}'$ for which $U = TU'$.

Theorem. *Algebraic semantics has an adjoint* $\hat{\mathfrak{S}}: \mathscr{K} \to \mathscr{T}^*$ *(which we call* algebraic structure), *and furthermore* $\mathfrak{S}\hat{\mathfrak{S}}$ *is naturally equivalent to the identity functor on* $\mathscr{T}^*$. *Explicitly, for any $U$ in $\mathscr{K}$ the $n$-ary operations of the algebraic theory* $U\hat{\mathfrak{S}}$ *are the natural transformations* $U^n \to U$.

Thus, any category $\mathscr{X}$ equipped with an "underlying set functor" $U$ determines an algebraic category $\mathscr{S}^{(U\hat{\mathfrak{S}})}$ together with a functor: $\Phi: \mathscr{X} \to \mathscr{S}^{(U\hat{\mathfrak{S}})}$ which preserves underlying sets, and given any other such functor $\Psi: \mathscr{X} \to \mathscr{S}^{(\mathbf{A})}$, there is a unique: $f: \mathbf{A} \to U\hat{\mathfrak{S}}$ in $\mathscr{T}$ such that $\Psi = \Phi\mathscr{S}^{(f)}$. Also the operations which define an algebraic category are in natural one-to-one correspondence with the natural operations on its underlying set functor. This second assertion of the theorem is a consequence of Yoneda's lemma (Yoneda [54]) and the fact that $U_{\mathbf{A}}^n = \mathrm{Hom}(A_n, ?)$ where $A_n$ is a free $\mathbf{A}$-algebra on $n$ generators. Thus the functor $\hat{\mathfrak{S}}$ becomes most interesting when applied to "underlying set functors" on non-algebraic categories. For example, if we take for $\mathscr{X}$ the dual of the category of sets and for $U$ the (contravariant) power set functor, the algebraic structure of $U$ is the theory of Boolean algebras, and $\Phi$ assigns to each set the Boolean algebra of its subsets. If $\mathscr{X}$ is the category of models for some given first-order theory $T$ (maps in $\mathscr{X}$ are elementary embeddings) and if $U$ is the "universe" functor, then the set of $n$-ary operations of the algebraic theory $U\hat{\mathfrak{S}}$ is identified as the set of all formulas of $T$ with at most $v_0, \ldots, v_n$ free which provably express $v_0$ as an everywhere well-defined function of $v_1, \ldots, v_n$, reduced modulo provable equivalence. If the theory $T$ can be logically generated by formulas of this type, then the functor $\Phi$ may be thought of as an inclusion; in general $\Phi$ measures semantically the departure of $T$ from its nearest "algebraic approximation", roughly speaking.

The above theorem implies that a category $\mathscr{X}$ is equivalent to some algebraic category iff it has some underlying set functor $U$ such that the particular functor $\Phi : \mathscr{X} \to \mathscr{S}^{(\mathbf{A})}$ described above is an equivalence where $\mathbf{A} = U\hat{\mathfrak{S}}$. In that case we must actually have $U = \mathrm{Hom}(G, ?)$ where $G$ is an object in $\mathscr{X}$ such that $G\Phi$ is a free $\mathbf{A}$-algebra on one generator. These observations enable us to completely characterize algebraic categories in the theorem below. We first define some terms.

By an *equalizer* and *coequalizer* of a pair of maps

$$X \underset{f_1}{\overset{f_0}{\rightrightarrows}} Y$$

in a category $\mathscr{X}$, we mean respectively an inverse limit $j: K \to X$ and a direct limit $p: Y \to K^*$ of the above diagram in the sense of Kan [58]. A map $p$ is called a *regular epimap* iff it is a coequalizer of some pair of maps. A category *has finite limits* iff it has an initial and a final object and if all possible binary products and coproducts (=free products), as well as all possible equalizers and coequalizers, exist. A pair $f_0, f_1$ as above is called a *precongruence* iff the corresponding $f: X \to Y \times Y$ is a monomorphism as well as reflexive, symmetric, and transitive in an obvious sense; the pair is called a *congruence* iff $f$ is the equalizer of the pair $\pi_0 p, \pi_1 p$, where $\pi_i: Y \times Y \to Y$ are the projections, and $p$ is the coequalizer of $f_0, f_1$. In a category with finite limits, every congruence is a precongruence, while in any algebraic category the converse is also true.

An object $G$ in a category $\mathscr{X}$ is *abstractly finite* iff the following three conditions hold: (a) for every object $X$, the class $\mathrm{Hom}(G, X)$ of maps $G \to X$ is small (i.e., is a "set"); (b) for every set $I$, the $I$-fold coproduct $I \cdot G$ esists; (c) any map $G \to I \cdot G$ factors through some $I' \cdot G$, where $I' \subseteq I$ and $I'$ is finite. A map $f: Y \to Z$ in $\mathscr{X}$ is *$G$-surjective* iff every map $G \to Z$ factors across $f$. An object $G$ is a *regular projective generator* iff the $G$-surjective maps are precisely the regular epimaps.

Finally, a functor $\Phi$ is said to be *full and faithful* iff for every pair $X$, $Y$ of objects in its domain category, $\Phi$ induces a one-to-one correspondence of the set of maps $X \to Y$ onto the set of maps $X\Phi \to Y\Phi$ in the codomain category.

Theorem. *Let $\mathscr{X}$ be a category with the following properties:* (0) *$\mathscr{X}$ has finite limits;* (1) *$\mathscr{X}$ has an abstractly finite regular projective generator $G$. Then there is an algebraic theory $\mathbf{A}$ and a functor $\Phi: \mathscr{X} \to \mathscr{S}^{(\mathbf{A})}$ which is full, faithful, and has an adjoint, and the free objects in $\mathscr{X}$ coincide with those in $\mathscr{S}^{(\mathbf{A})}$ (i.e., $(I \cdot G)\Phi = I \cdot (G\Phi)$). Furthermore $\Phi$ is an equivalence iff* (2) *every precongruence in $\mathscr{X}$ is a congruence. Conditions* 0, 1, 2 *are necessary and sufficient that $\mathscr{X}$ be equivalent to some algebraic category.*

For the proof, choose $G$ as guaranteed by condition 1, set $U = \mathrm{Hom}(G, ?)$ and let $\mathbf{A} = U\hat{\mathfrak{S}}$. Then by the previous theorem, there is a functor $\Phi: \mathscr{X} \to \mathscr{S}^{(\mathbf{A})}$ with a universal mapping property. By using the properties of $G$, we can show that $\Phi$ is full and faithful and preserves coproducts of $G$ with itself. Also (as was pointed out by Isbell [a]) every sub-$\mathbf{A}$-algebra of any $X\Phi$ is also of the form $X'\Phi$. To construct the adjoint to $\Phi$, consider a presentation of an $\mathbf{A}$-algebra $Y$ as a quotient of a free $\mathbf{A}$-algebra; as the associated congruence relation actually comes from $\mathscr{X}$ we may perform the "same" quotient operation in $\mathscr{X}$ to obtain $\overline{Y}$, the value at $Y$ of the adjoint of $\Phi$. That $\overline{Y}\Phi \simeq Y$ need not hold is shown by the following example: $\mathscr{X}$ = category of torsion-free abelian groups, $G$ = rational integers, where $\overline{Y}$ is the result of reducing the arbitrary abelian group $Y$ modulo its torsion subgroup. However, if condition 2 also holds, then the adjoint of $\Phi$ is (up to natural equivalence) actually inverse to $\Phi$.

Corollary. *If $\mathscr{X}$ is an algebraic category and if $\mathscr{C}$ is any small category with finitely many objects, then the full category $\mathscr{X}^{\mathscr{C}}$ of functors $\mathscr{C} \to \mathscr{X}$*

*and natural transformations thereof is itself equivalent to an algebraic category, provided either that* $\mathscr{X} \cong \mathscr{S}^{(\mathbf{A})}$ *where* $\mathbf{A}$ *has at least one nullary operation or that for any two objects* $C$, $C^1$ *in* $\mathscr{C}$, *there exists a map* $C \rightarrow C^1$ *in* $\mathscr{C}$.

In proving this corollary, we define the underlying set of a functor $\mathscr{C} \xrightarrow{F} \mathscr{X}$ to be the product, indexed by the objects $C$ in $\mathscr{C}$, of the sets $CFU$, where $U$ is a given underlying set functor for $\mathscr{X}$. When $\mathscr{C}$ has only one object, then $\mathscr{C}$ is a monoid and the corollary refers to the well-known notion of a monoid acting by endomorphisms of an algebra to form a new type of algebra.